Chapter 9

Applications of Integration

Many physical and geometric quantities can be expressed as integrals.

Our applications of integration in Chapter 4 were limited to area, distance–velocity, and rate problems. In this chapter, we will see how to use integrals to set up problems involving volumes, averages, centers of mass, work, energy, and power. The techniques developed in Chapter 7 make it possible to solve many of these problems completely.

9.1 Volumes by the Slice Method

The volume of a solid region is an integral of its cross-sectional areas.

By thinking of a region in space as being composed of "infinitesimally thin slices," we shall obtain a formula for volumes in terms of the areas of slices. In this section, we apply the formula in a variety of special cases. Further methods for calculating volumes will appear when we study multiple integration in Chapter 17.

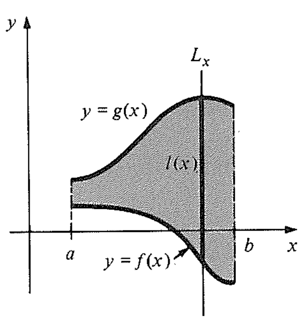


Figure 9.1.1. The area of the shaded region is $\int_a^b l(x)\,dx$.

We will develop the slice method for volumes by analogy with the computation of areas by integration. If f and g are functions with $f(x) \leqslant g(x)$ on $[a,b]$, then the area between the graphs of f and g is $\int_a^b [g(x) - f(x)]\,dx$ (see Section 4.6). We recall the infinitesimal argument for this formula. Think of the region as being composed of infinitesimally thin strips obtained by cutting with lines perpendicular to the x axis. Denote the vertical line through x by L_x; the intersection of L_x with the region between the graphs has length $l(x) = g(x) - f(x)$, and the corresponding "infinitesimal rectangle" with thickness dx has area $l(x)\,dx$ (= height × width) (see Fig. 9.1.1). The area of the entire region, obtained by "summing" the infinitesimal areas, is

$$\int_a^b l(x)\,dx = \int_a^b [\,g(x) - f(x)\,]\,dx.$$

Given a region surrounded by a closed curve, we can often use the same formula $\int_a^b l(x)\,dx$ to find its area. To implement this, we position it conveniently with respect to the axes and determine a and b by noting where the ends of the region are. We determine $l(x)$ by using the geometry of the situation at hand. This is done for a disk of radius r in

Fig. 9.1.2. We may evaluate the integral $\int_{-r}^{r} 2\sqrt{r^2 - x^2}\,dx$ by using integral tables to obtain the answer πr^2, in agreement with elementary geometry. (One can also readily evaluate integrals of this type by using the substitution $x = r\cos\theta$.)

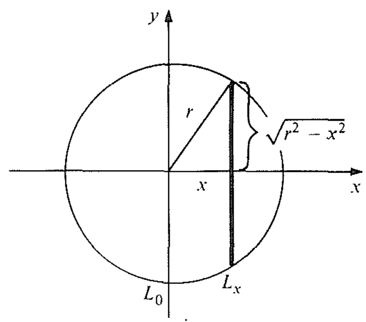


Figure 9.1.2. Area of the disk $= \int_{-r}^{r} 2\sqrt{r^2 - x^2}\,dx$.

To find the volume of a *solid* region, we imagine it sliced by a family of parallel *planes*: The plane P_x is perpendicular to a fixed x axis in space at a distance x from a reference point (Fig. 9.1.3).

The plane P_x cuts the solid in a plane region R_x; the corresponding "infinitesimal piece" of the solid is a slab whose base is a region R_x and whose thickness is dx (Fig. 9.1.4). The volume of such a cylinder is equal to the area of the base R_x times the thickness dx. If we denote the area of R_x by $A(x)$, then this volume is $A(x)\,dx$. Thus the volume of the entire solid, obtained by summing, is the integral $\int_a^b A(x)\,dx$, where the limits a and b are determined by the ends of the solid.

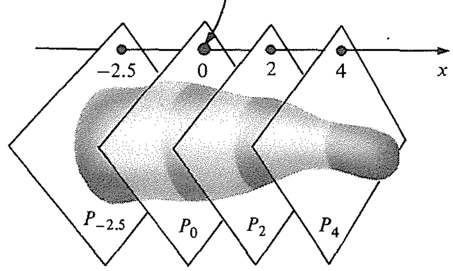


Figure 9.1.3. The plane P_x is at distance x from P_0.

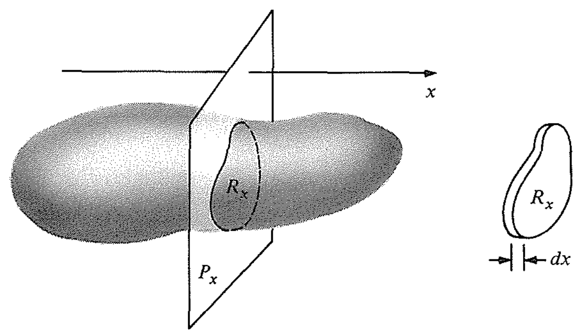


Figure 9.1.4. An infinitesimally thin slice of a solid.

The Slice Method

Let S be a solid and P_x be a family of parallel planes such that:

1. S lies between P_a and P_b;
2. the area of the slice of S cut by P_x is $A(x)$.

Then the volume of S is equal to

$$\int_a^b A(x)\,dx.$$

The slice method can also be justified using step functions. We shall see how to do this below.

In simple cases, the areas $A(x)$ can be computed by elementary geometry. For more complicated problems, it may be necessary to do a preliminary integration to find the $A(x)$'s themselves.

Example 1 Find the volume of a ball[1] of radius r.

Solution Draw the ball above the x axis as in Fig. 9.1.5.

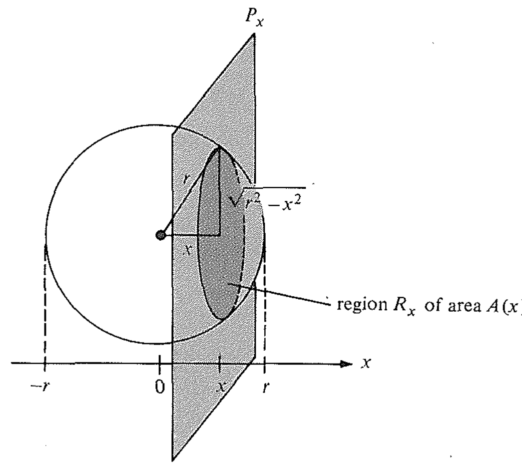


Figure 9.1.5. The area of the slice at x of a ball of radius r is $A(x) = \pi(r^2 - x^2)$.

Let the plane P_0 pass through the center of the ball. The ball lies between P_{-r} and P_r, and the slice R_x is a disk of radius $\sqrt{r^2 - x^2}$. The area of the slice is $\pi \times (\text{radius})^2$; i.e., $A(x) = \pi(\sqrt{r^2 - x^2}\,)^2 = \pi(r^2 - x^2)$. Thus the volume is

$$\int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = \pi\left(r^2x - \frac{x^3}{3}\right)\Bigg|_{-r}^{r} = \frac{4}{3}\pi r^3. \blacktriangle$$

Example 2 Find the volume of the conical solid in Fig. 9.1.6. (The base is a circle.)

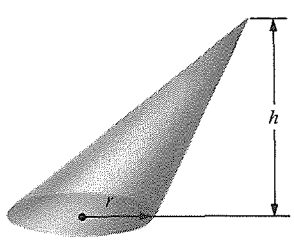
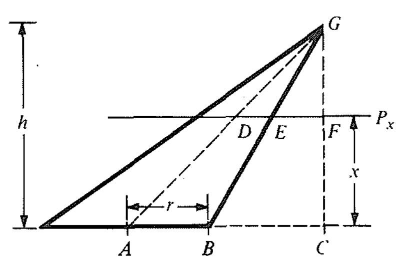


Figure 9.1.6. Find the volume of this oblique circular cone.

Figure 9.1.7. $|DE|/|AB| = |GE|/|GB| = |GF|/|GC|$ by similar triangles. But $|AB| = r$, $|GC| = h$, and $|GF| = h - x$, and so $|DE| = [(h - x)/h]r$.



Solution We let the x axis be vertical and choose the family P_x of planes such that P_0 contains the base of the cone and P_x is at distance x above P_0. Then the cone lies between P_0 and P_h, and the plane section by P_x is a disk with radius $[(h - x)/h]r$ and area $\pi[(h - x)/h]^2r^2$ (see Fig. 9.1.7). By the slice method,

[1] A *sphere* is the set of points in space at a fixed distance from a point. A *ball* is the solid region enclosed by a sphere, just as a disk is the plane region enclosed by a circle.

the volume is

$$\int_0^h A(x)\,dx = \int_0^h \pi \frac{(h-x)^2}{h^2} r^2\,dx = \frac{\pi r^2}{h^2}\int_0^h (h^2 - 2xh + x^2)\,dx$$

$$= \frac{\pi r^2}{h^2}\left[(h^2x - hx^2) + \frac{x^3}{3}\right]\Bigg|_0^h = \frac{1}{3}\pi r^2 h. \blacktriangle$$

Example 3 Find the volume of the solid W shown in Fig. 9.1.8. It can be thought of as a wedge-shaped piece of a cylindrical tree of radius r obtained by making two saw cuts to the tree's center, one horizontally and one at an angle θ.

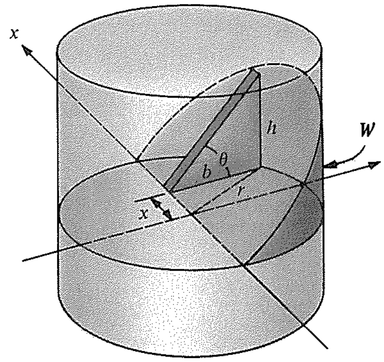


Figure 9.1.8. Find the volume of the wedge W.

Solution With the setup in Fig. 9.1.8, we slice W by planes to produce triangles R_x of area $A(x)$ as shown. The base b of the triangle is $b = \sqrt{r^2 - x^2}$, and its height is $h = b\tan\theta = \sqrt{r^2 - x^2}\tan\theta$. Thus, $A(x) = \frac{1}{2}bh = \frac{1}{2}(r^2 - x^2)\tan\theta$. Hence, the volume is

$$\int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \frac{1}{2}(r^2 - x^2)\tan\theta\,dx = \frac{1}{2}(\tan\theta)\left(r^2x - \frac{x^3}{3}\right)\Bigg|_{-r}^{r}$$

$$= \frac{1}{2}(\tan\theta)\left(2r^3 - \frac{2r^3}{3}\right) = \frac{2r^3}{3}\tan\theta.$$

Notice that even though we started with a region with a circular boundary, π does not occur in the answer! $\blacktriangle$

Example 4 A ball of radius r is cut into three pieces by parallel planes at a distance of $r/3$ on each side of the center. Find the volume of each piece.

Solution The middle piece lies between the planes $P_{-r/3}$ and $P_{r/3}$ of Example 1, and the area function is $A(x) = \pi(r^2 - x^2)$ as before, so the volume of the middle piece is

$$\int_{-r/3}^{r/3} \pi(r^2 - x^2)\,dx = \pi\left(r^2x - \frac{x^3}{3}\right)\Bigg|_{-r/3}^{r/3}$$

$$= \pi\left(\frac{r^3}{3} - \frac{r^3}{81} + \frac{r^3}{3} - \frac{r^3}{81}\right) = \frac{52}{81}\pi r^3.$$

This leaves a volume of $(\frac{4}{3} - \frac{52}{81})\pi r^3 = \frac{56}{81}\pi r^3$ to be divided between the two outside pieces. Since they are congruent, each of them has volume $\frac{28}{81}\pi r^3$. (You may check this by computing $\int_{r/3}^{r}\pi(r^2 - x^2)\,dx$.) $\blacktriangle$

One way to construct a solid is to take a plane region R, as shown in Fig. 9.1.9 and revolve it around the x axis so that it sweeps out a solid region S. Such solids are common in woodworking shops (lathe-tooled table legs), in pottery

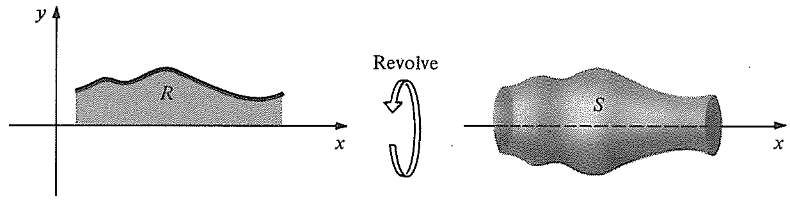


Figure 9.1.9. S is the solid of revolution obtained by revolving the plane region R about the x axis.

studios (wheel-thrown pots), and in nature (unicellular organisms).[2] They are called *solids of revolution* and are said to have *axial symmetry*.

Suppose that region R is bounded by the lines $x = a$, $x = b$, and $y = 0$, and by the graph of the function $y = f(x)$. To compute the volume of S by the slice method, we use the family of planes perpendicular to the x axis, with P_0 passing through the origin. The plane section of S by P_x is a circular disk of radius $f(x)$ (see Fig. 9.1.10), so its area $A(x)$ is $\pi[f(x)]^2$. By the basic formula of the slice method, the volume of S is

$$\int_a^b A(x)\,dx = \int_a^b \pi[f(x)]^2\,dx = \pi\int_a^b [f(x)]^2\,dx.$$

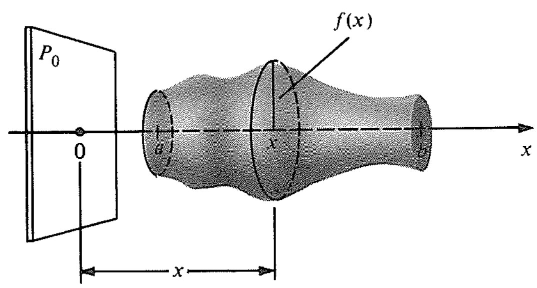


Figure 9.1.10. The volume of a solid of revolution obtained by the disk method.

We use the term "disk method" for this special case of the slice method since the slices are disks.

Volume of a Solid of Revolution: Disk Method

The volume of the solid of revolution obtained by revolving the region under the graph of a (non-negative) function $f(x)$ on $[a,b]$ about the x axis is

$$\pi\int_a^b [f(x)]^2\,dx.$$

Example 5 The region under the graph of x^2 on $[0,1]$ is revolved about the x axis. Sketch the resulting solid and find its volume.

Solution The solid, which is shaped something like a trumpet, is sketched in Fig. 9.1.11.

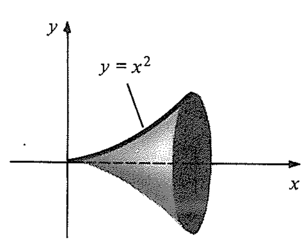


Figure 9.1.11. The volume of this solid of revolution is $\pi\int_0^1 (x^2)^2\,dx$.

[2] See D'Arcy Thompson, *On Growth and Form*, abridged edition, Cambridge University Press (1969).

According to the disk method, its volume is

$$\pi\int_0^1 (x^2)^2\,dx = \pi\int_0^1 x^4\,dx = \frac{\pi x^5}{5}\Big|_0^1 = \frac{\pi}{5}. \ \blacktriangle$$

Example 6 The region between the graphs of $\sin x$ and x on $[0,\pi/2]$ is revolved about the x axis. Sketch the resulting solid and find its volume.

Solution The solid is sketched in Fig. 9.1.12. It has the form of a hollowed-out cone.

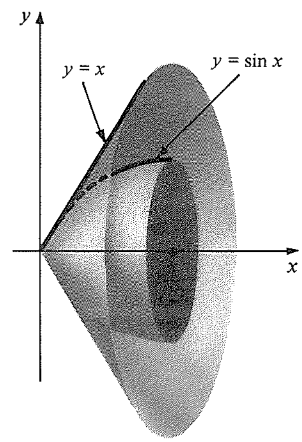


Figure 9.1.12. The region between the graphs of $\sin x$ and x is revolved about the x axis.

The volume is that of the cone minus that of the hole. The cone is obtained by revolving the region under the graph of x on $[0,1]$ about the axis, so its volume is

$$\pi\int_0^{\pi/2} x^2\,dx = \frac{\pi^4}{24}.$$

The hole is obtained by revolving the region under the graph of $\sin x$ on $[0,\pi/2]$ about the x axis, so its volume is

$$\pi\int_0^{\pi/2}\sin^2 x\,dx = \pi\int_0^{\pi/2}\frac{1-\cos 2x}{2}\,dx \qquad (\text{since } \cos 2x = 1-2\sin^2 x)$$

$$= \pi\left(\frac{x}{2}-\frac{1}{4}\sin 2x\right)\Big|_0^{\pi/2}$$

$$= \pi\left(\frac{\pi}{4}-0-0+0\right) = \frac{\pi^2}{4}.$$

Thus the volume of our solid is $\pi^4/24-\pi^2/4 \approx 1.59$. $\blacktriangle$

The volume of the solid obtained by rotating the region between the graphs of two functions f and g (with $f(x) \leqslant g(x)$ on $[a,b]$) can be done as in Example 6 or by the *washer method* which proceeds as follows. In Fig. 9.1.13, the volume of the shaded region (the "washer") is the area $\times$ thickness. The area of the washer is the area of the complete disk minus that of the hole. Thus, the washer's volume is

$$\left(\pi[g(x)]^2-\pi[f(x)]^2\right)dx.$$

Thus, the total volume is

$$\pi\int_a^b\left([g(x)]^2-[f(x)]^2\right)dx.$$

The reader should notice that this method gives the same answer as one finds by using the method of Example 6.

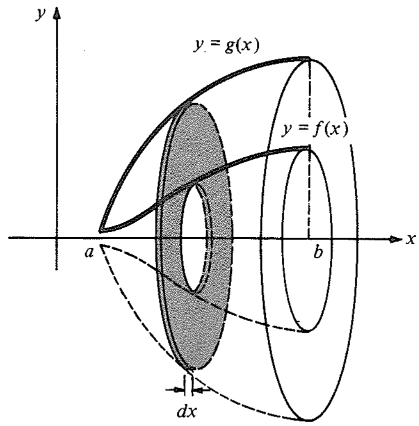


Figure 9.1.13. The washer method.

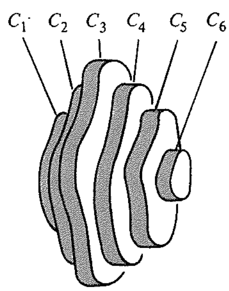


Figure 9.1.14. A "stepwise cylindrical" solid.

Our formula for volumes by the slice method was introduced via infinitesimals. A more rigorous argument for the formula is based on the use of upper and lower sums.[3] To present this argument, we first look at the case where S is composed of n cylinders, as in Fig. 9.1.14.

If the ith cylinder C_i lies between the planes $P_{x_{i-1}}$ and P_{x_i} and has cross-sectional area k_i, then the function $A(x)$ is a step function on the interval $[x_0, x_n]$; in fact, $A(x) = k_i$ for x in (x_{i-1}, x_i). The volume of C_i is the product of its base area k_i by its height $\Delta x_i = x_i - x_{i-1}$, so the volume of the total figure is $\sum_{i=1}^{n} k_i \Delta x_i$; but this is just the integral $\int_{x_0}^{x_n} A(x)\,dx$ of the step function $A(x)$. We conclude that if S is a "stepwise cylindrical" solid between the planes P_a and P_b, then

$$\text{volume } S = \int_a^b A(x)\,dx.$$

If S is a reasonably "smooth" solid region, we expect that it can be squeezed arbitrarily closely between stepwise cylindrical regions on the inside and outside. Specifically, for every positive number ε, there should be a stepwise cylindrical region S_i inside S and another such region S_o outside S such that (volume S_o) $-$ (volume S_i) $< \varepsilon$. If $A_i(x)$ and $A_o(x)$ are the corresponding functions, then A_i and A_o are step functions, and we have the inequality $A_i(x) \leqslant A(x) \leqslant A_o(x)$, so

$$\text{volume } S_i = \int_a^b A_i(x)\,dx \leqslant \int_a^b A(x)\,dx \leqslant \int_a^b A_o(x)\,dx = \text{volume } S_o.$$

Since S encloses S_i and S_o encloses S, volume $S_i \leqslant$ volume $S \leqslant$ volume S_o. Thus the numbers (volume S) and $\int_a^b A(x)\,dx$ both belong to the same interval [(volume S_i), (volume S_o)], which has length less than ε. It follows that the difference between (volume S) and $\int_a^b A(x)\,dx$ is less than any positive number ε; the only way this can be so is if the two numbers are equal.

Supplement to Section 9.1: Cavalieri's Delicatessen

The idea behind the slice method goes back, beyond the invention of calculus, to Francesco Bonaventura Cavalieri (1598–1647), a student of Galileo and then professor at the University of Bologna. An accurate report of the events leading to Cavalieri's discovery is not available, so we have taken the liberty of inventing one.

[3] Even this justification, as we present it, is not yet completely satisfactory. For example, do we get the same answer if we slice the solid a different way? The answer is yes, but the proof uses multiple integrals (see Chapter 17).

Cavalieri's delicatessen usually produced bologna in cylindrical form, so that the volume would be computed as $\pi \cdot \text{radius}^2 \cdot \text{length}$. One day, the casings were a bit weak, and the bologna came out with odd bulges. The scale was not working that day, either, so the only way to compute the price of the bologna was in terms of its volume.

Cavalieri took his best knife and sliced the bologna into n very thin slices, each of thickness Δx, and measured the radii $r_1, r_2, \dots, r_n$ of the slices (fortunately, they were round). He then estimated the volume to be $\sum_{i=1}^{n} \pi r_i^2 \Delta x_i$, the sum of the volumes of the slices.

Cavalieri was moonlighting from his regular job as a professor at the University of Bologna. That afternoon, he went back to his desk and began the book "Geometria indivisibilium continuorum nova quandum ratione promota" ("Geometry shows the continuous indivisibility between new rations and getting promoted"), in which he stated what is now known[4] as *Cavalieri's principle*:

> *If two solids are sliced by a family of parallel planes in such a way that corresponding sections have equal areas, then the two solids have the same volume.*

The book was such a success that Cavalieri sold his delicatessen and retired to a life of occasional teaching and eternal glory.

[4] Honest!

Exercises for Section 9.1

In Exercises 1–4, use the slice method to find the volume of the indicated solid.

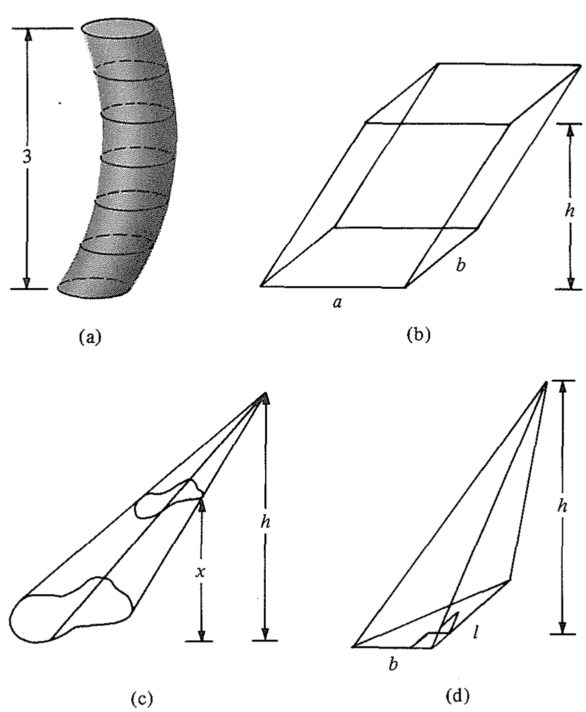


Figure 9.1.15. The solids for Exercises 1–4.

1. The solid in Fig. 9.1.15(a); each plane section is a circle of radius 1.
2. The parallelepiped in Fig. 9.1.15(b); the base is a rectangle with sides a and b.
3. The solid in Fig. 9.1.15(c); the base is a figure of area A and the figure at height x has area $A_x = [(h-x)/h]^2 A$.
4. The solid in Fig. 9.1.15(d); the base is a right triangle with sides b and l.
5. Find the volume of the tent in Fig. 9.1.16. The plane section at height x above the base is a square of side $\frac{1}{6}(6-x)^2 - \frac{1}{6}$. The height of the tent is 5 feet.

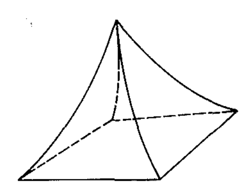


Figure 9.1.16. Find the volume of this tent.

6. What would the volume of the tent in the previous exercise be if the base and cross sections were equilateral triangles instead of squares (with the same side lengths)?
7. The base of a solid S is the disk in the xy plane with radius 1 and center $(0,0)$. Each section of S cut by a plane perpendicular to the x axis is an equilateral triangle. Find the volume of S.

8. A plastic container is to have the shape of a truncated pyramid with upper and lower bases being squares of side length 10 and 6 centimeters, respectively. How high should the container be to hold exactly one liter (= 1000 cubic centimeters)?
9. The conical solid in Fig. 9.1.6 is to be cut by horizontal planes into four pieces of equal volume. Where should the cuts be made? [*Hint:* What is the volume of the portion of the cone above the plane P_x?]
10. The tent in Exercise 5 is to be cut into two pieces of equal volume by a plane parallel to the base. Where should the cut be made?
    (a) Express your answer as the root of a fifth-degree polynomial.
    (b) Find an approximate solution using the method of bisection.
11. A wedge is cut in a tree of radius 0.5 meter by making two cuts to the tree's center, one horizontal and another at an angle of 15° to the first. Find the volume of the wedge.
12. A wedge is cut in a tree of radius 2 feet by making two cuts to the tree's center, one horizontal and another at an angle of 20° to the first. Find the volume of the wedge.
13. Find the volume of the solid in Fig. 9.1.17(a).
14. Find the volume of the solid in Fig. 9.1.17(b).

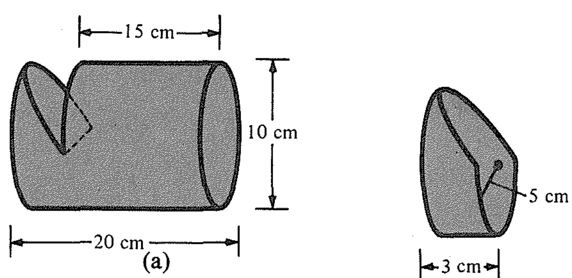


Figure 9.1.17. Find the volumes of these solids.

In Exercises 15–26, find the volume of the solid obtained by revolving each of the given regions about the x axis and sketch the region.

15. The region under the graph of $3x + 1$ on $[0, 2]$.
16. The region under the graph of $2 - (x - 1)^2$ on $[0, 2]$.
17. The region under the graph of $\cos x + 1$ on $[0, 2\pi]$.
18. The region under the graph of $\cos 2x$ on $[0, \pi/4]$.
19. The region under the graph of $x(x - 1)^2$ on $[1, 2]$.
20. The region under the graph of $\sqrt{4 - 4x^2}$ on $[0, 1]$.
21. The semicircular region with center $(a, 0)$ and radius r (assume that $0 < r < a$, $y \geqslant 0$).
22. The region between the graphs of $\sqrt{3 - x^2}$ and $5 + x$ on $[0, 1]$. (Evaluate the integral using geometry or the tables.)
23. The square region with vertices $(4, 6)$, $(5, 6)$, $(5, 7)$, and $(4, 7)$.
24. The region in Exercise 23 moved 2 units upward.
25. The region in Exercise 23 rotated by 45° around its center.
26. The triangular region with vertices $(1, 1)$, $(2, 2)$, and $(3, 1)$.

★27. A vase with axial symmetry has the cross section shown in Fig. 9.1.18 when it is cut by a plane through its axis of symmetry. Find the volume of the vase to the nearest cubic centimeter.

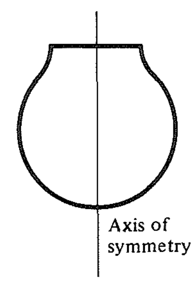


Figure 9.1.18. Cross section of a vase.

★28. A right circular cone of base radius r and height 14 is to be cut into three equal pieces by parallel planes which are parallel to the base. Where should the cuts be made?
★29. Find the formula for the volume of a doughnut with outside radius R and a hole of radius r.
★30. Use the fact that the area of a disk of radius r is $\pi r^2 = \int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx$ to compute the area inside the ellipse $y^2/4 + x^2 = r^2$.
★31. Prove Cavalieri's principle.
★32. Using Cavalieri's principle, without integration, find a relation between the volumes of:
    (a) a hemisphere of radius 1;
    (b) a right circular cone of base radius 1 and height 1;
    (c) a right circular cylinder of base radius 1 and height 1.

    [*Hint:* Consider two of the solids side by side as a single solid. The sum of two volumes will equal the third.]

9.2 Volumes by the Shell Method

A solid of revolution about the y axis can be regarded as composed of cylindrical shells.

In the last section, we computed the volume of the solid obtained by revolving the region under the graph of a function about the x axis. Another way to obtain a solid S is to revolve the region R under the graph of a non-negative function $f(x)$ on $[a,b]$ about the y axis as shown in Fig. 9.2.1. We assume that $0 \leqslant a < b$.

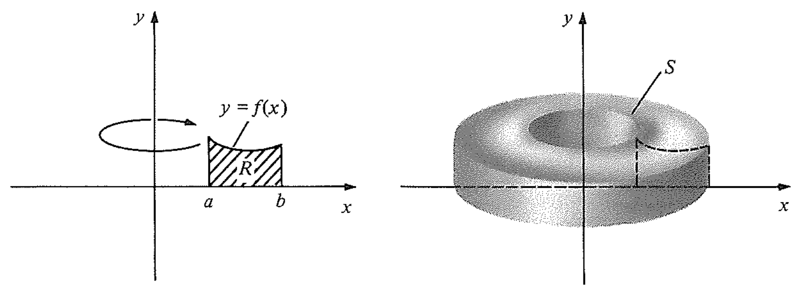


Figure 9.2.1. The solid S is obtained by revolving the plane region R about the y axis.

To find the volume of S, we use the method of infinitesimals. (Another argument using step functions is given at the end of the section.) If we rotate a strip of width dx and height $f(x)$ located at a distance x from the axis of rotation, the result is a cylindrical shell of radius x, height $f(x)$, and thickness dx. We may "unroll" this shell to get a flat rectangular sheet whose length is $2\pi x$, the circumference of the cylindrical shell (see Fig. 9.2.2). The volume of the sheet is thus the product of its area $2\pi x f(x)$ and its thickness dx. The total volume of the solid, obtained by summing the volumes of the infinitesimal shells, is the integral $\int_a^b 2\pi x f(x)\,dx$. If we revolve the region between the graphs of $f(x)$ and $g(x)$, with $f(x) \leqslant g(x)$ on $[a,b]$, the height is $g(x) - f(x)$, and so the volume is $2\pi\int_a^b x[g(x) - f(x)]\,dx$.

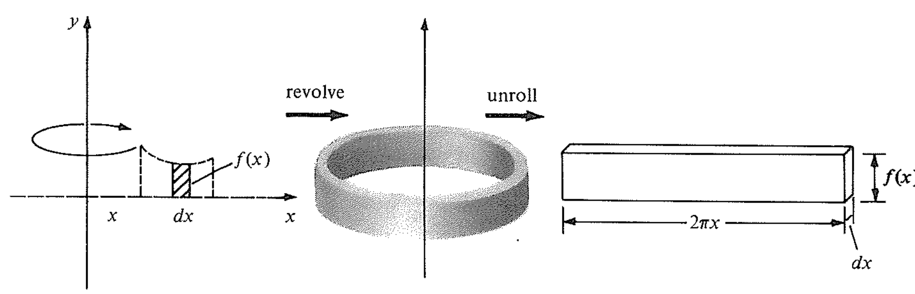


Figure 9.2.2. The volume of the cylindrical shell is $2\pi x f(x)\,dx$.

Example 1 The region under the graph of x^2 on $[0,1]$ is revolved about the y axis. Sketch the resulting solid and find its volume.

Solution The solid, in the shape of a bowl, is sketched in Fig. 9.2.3. Its volume is

$$2\pi\int_0^1 x\cdot x^2\,dx = 2\pi\int_0^1 x^3\,dx = 2\pi\,\frac{x^4}{4}\bigg|_0^1 = \frac{\pi}{2}\,. \ \blacktriangle$$

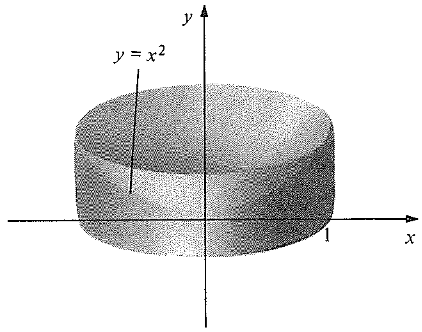


Figure 9.2.3. Find the volume of the "bowl-like" solid.

> **Volume of a Solid of Revolution: Shell Method**
>
> The volume of the solid of revolution obtained by revolving about the y axis the region under the graph of a (non-negative) function $f(x)$ on $[a,b]$ $(0 \leqslant a < b)$ is
>
> $$2\pi\int_a^b xf(x)\,dx.$$
>
> If the region between the graphs of $f(x)$ and $g(x)$ is revolved, the volume is
>
> $$2\pi\int_a^b x[\,g(x)-f(x)\,]\,dx.$$

Example 2 Find the capacity of the bowl in Example 1.

Solution The capacity of the bowl is the volume of the region obtained by rotating the region between the curves $y = x^2$ and $y = 1$ on $[0, 1]$ around the y axis.

By the second formula in the box above, with $f(x) = x^2$ and $g(x) = 1$, the volume is

$$2\pi\int_0^1 x(1-x^2)\,dx = 2\pi\int_0^1 (x-x^3)\,dx = 2\pi\left(\frac{x^2}{2}-\frac{x^4}{4}\right)\Big|_0^1 = \frac{\pi}{2}\,. \ \blacktriangle$$

We could have found the capacity in Example 2 by subtracting the result of Example 1 from the volume of the right circular cylinder with radius 1 and height 1, namely $\pi r^2 h = \pi$. Another way to find the capacity is by the slice method, using y as the independent variable. The slice at height y is a disk of radius $x = \sqrt{y}$, so the volume is

$$\int_0^1 \pi\left(\sqrt{y}\right)^2 dy = \int_0^1 \pi y\,dy = \frac{1}{2}\pi y^2\Big|_0^1 = \frac{\pi}{2}\,.$$

Example 3 Sketch and find the volume of the solid obtained by revolving each of the following regions about the y axis: (a) the region under the graph of e^x on $[1,3]$; (b) the region under the graph of $2x^3 + 5x + 1$ on $[0,1]$.

Solution (a) Volume $= 2\pi\int_1^3 xe^x\,dx$. This integral may be evaluated by integration by parts to give $2\pi(xe^x|_1^3 - \int_1^3 e^x\,dx) = 2\pi[e^x(x-1)]|_1^3 = 4\pi e^3$. (see Fig. 9.2.4(a)).

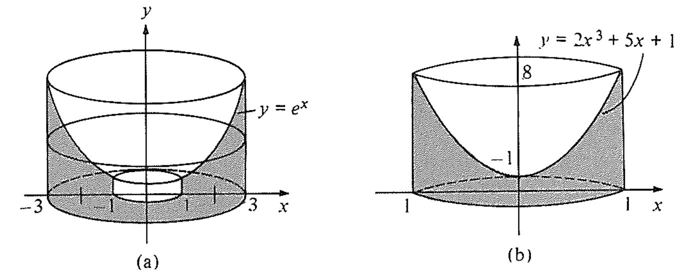


Figure 9.2.4. Find the volume of the shaded solids.

(b) Volume $= 2\pi\int_0^1 x(2x^3+5x+1)\,dx = 2\pi\left[\frac{2x^5}{5}+\frac{5x^3}{3}+\frac{x^2}{2}\right]\Big|_0^1 = \frac{77}{15}\pi.$

(See Fig. 9.2.4(b)). ▲

Example 4 Find the volume of the "flying saucer" obtained by rotating the region between the curves $y = -\frac{1}{4}(1-x^4)$ and $y = \frac{1}{6}(1-x^6)$ on $[0,1]$ about the y axis.

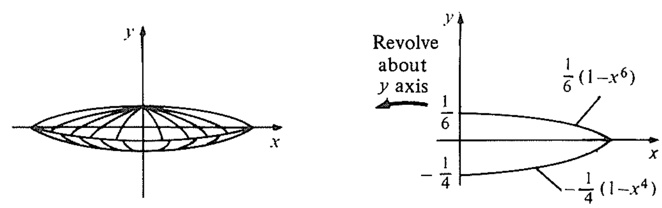


Figure 9.2.5. The flying saucer.

Solution See Fig. 9.2.5. The height of the shell at radius x is $\frac{1}{6}(1-x^6)+\frac{1}{4}(1-x^4)$ $= (5/12)-(x^6/6)-(x^4/4)$, so the volume is

$$2\pi\int_0^1 x\left(\frac{5}{12}-\frac{x^6}{6}-\frac{x^4}{4}\right)dx = 2\pi\left(\frac{5}{24}x^2-\frac{x^8}{48}-\frac{x^6}{24}\right)\Big|_0^1$$
$$= 2\pi\left(\frac{5}{24}-\frac{1}{48}-\frac{1}{24}\right)=\frac{7\pi}{24}.\ ▲$$

Example 5 A hole of radius r is drilled through the center of a ball of radius R. How much material is removed?

Solution See Figure 9.2.6. The shell at distance x from the axis of the hole has height

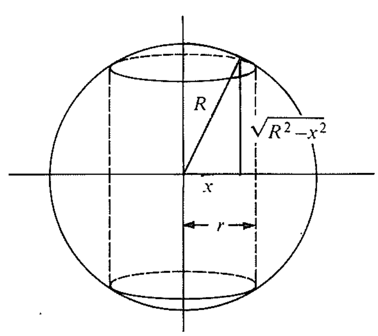


Figure 9.2.6. A ball with a hole drilled through it.

$2\sqrt{R^2-x^2}$. The shells removed have x running from 0 to r, so their total volume is

$$2\pi\int_0^r 2x\sqrt{R^2-x^2}\,dx = 2\pi\int_0^{r^2}\sqrt{R^2-u}\,du = 2\pi\left[-\frac{2}{3}(R^2-u)^{3/2}\Big|_0^{r^2}\right]$$
$$=\frac{4}{3}\pi\left[R^3-(R^2-r^2)^{3/2}\right]$$
$$=\frac{4}{3}\pi R^3\left[1-\left(1-\frac{r^2}{R^2}\right)^{3/2}\right].$$

Notice that if we set $r = R$, we get $\frac{4}{3}\pi R^3$; we then recover the formula for the volume of the ball, computed by the shell method. ▲

Example 6 The disk with radius 1 and center $(4,0)$ is revolved around the y axis. Sketch the resulting solid and find its volume.

Solution The doughnut-shaped solid is shown in Fig. 9.2.7.[5] We observe that if the solid is sliced in half by a plane through the origin perpendicular to the y axis, the top half is the solid obtained by revolving about the y axis the region under the semicircle $y=\sqrt{1-(x-4)^2}$ on the interval $[3,5]$.

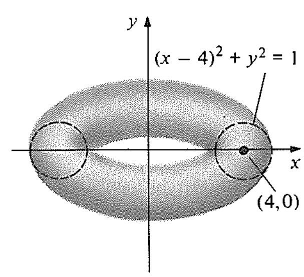


Figure 9.2.7. The disk $(x-4)^2+y^2 \leqslant 1$ is revolved about the y axis.

The volume of that solid is

$$2\pi\int_3^5 x\sqrt{1-(x-4)^2}\,dx$$
$$=2\pi\int_{-1}^{1}(u+4)\sqrt{1-u^2}\,du \qquad (u=x-4)$$
$$=2\pi\int_{-1}^{1}\sqrt{1-u^2}\,u\,du+8\pi\int_{-1}^{1}\sqrt{1-u^2}\,du.$$

Now $\int_{-1}^{1}\sqrt{1-u^2}\,u\,du=0$ because the function $f(u)=\sqrt{1-u^2}\,u$ is odd: $f(-u) = -f(u)$ so that $\int_{-1}^{0}f(u)\,du$ is exactly the negative of $\int_0^1 f(u)\,du$.

On the other hand, $\int_{-1}^{1}\sqrt{1-u^2}\,du$ is just the area of a semicircular region of radius 1—that is, $\pi/2$—so the volume of the upper half of the doughnut is $8\pi\cdot(\pi/2)=4\pi^2$, and the volume of the entire doughnut is twice that, or $8\pi^2$. (Notice that this is equal to the area π of the rotated disk times the circumference 8π of the circle traced out by its center $(4,0)$.) ▲

[5]Mathematicians call this a *solid torus*. The surface of this solid (an "inner tube") is a *torus*.

We conclude this section with a justification of the shell method using step functions. Consider again the solid S in Fig. 9.2.1. We break the region R into thin vertical strips and rotate them into shells, as in Fig. 9.2.8.

What is the volume of such a shell? Suppose for a moment that f has the

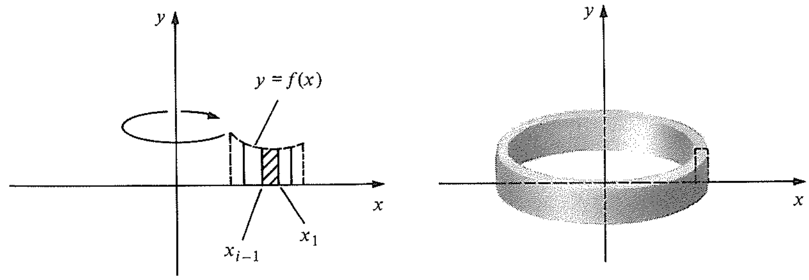


Figure 9.2.8. The volume of a solid of revolution obtained by the shell method.

constant value k_i on the interval (x_{i-1}, x_i). Then the shell is the "difference" of two cylinders of height k_i, one with radius x_i and one with radius x_{i-1}. The volume of the shell is, therefore, $\pi x_i^2 k_i - \pi x_{i-1}^2 k_i = \pi k_i(x_i^2 - x_{i-1}^2)$; we may observe that this last expression is $\int_{x_{i-1}}^{x_i} 2\pi k_i x\,dx$.

If f is a step function on $[a,b]$, with partition $(x_0, \ldots, x_n)$ and $f(x) = k_i$ on (x_{i-1}, x_i), then the volume of the collection of n shells is

$$\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} 2\pi k_i x\,dx;$$

but $k_i = f(x)$ on (x_{i-1}, x_i), so this is

$$\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} 2\pi x f(x)\,dx,$$

which is simply $\int_a^b 2\pi x f(x)\,dx$. We now have the formula

$$\text{volume} = 2\pi \int_a^b x f(x)\,dx,$$

which is valid whenever $f(x)$ is a step function on $[a,b]$. To show that the same formula is valid for general f, we squeeze f between step functions above and below using the same argument we used for the slice method.

Exercises for Section 9.2

In Exercises 1–12, find the volume of the solid obtained by revolving each of the following regions about the y axis and sketch the region.

1. The region under the graph of $\sin x$ on $[0,\pi]$.
2. The region under the graph of $\cos 2x$ on $[0,\pi/4]$.
3. The region under the graph of $2-(x-1)^2$ on $[0,2]$.
4. The region under the graph of $\sqrt{4-4x^2}$ on $[0,1]$.
5. The region between the graphs of $\sqrt{3-x^2}$ and $5+x$ on $[0,1]$.
6. The region between the graphs of $\sin x$ and x on $[0,\pi/2]$.
7. The circular region with center $(a,0)$ and radius r $(0<r<a)$.
8. The circular region with radius 2 and center $(6,0)$.
9. The square region with vertices $(4,6)$, $(5,6)$, $(5,7)$, and $(4,7)$.
10. The region in Exercise 9 moved 2 units upward.
11. The region in Exercise 9 rotated by 45° around its center.
12. The triangular region with vertices $(1,1)$, $(2,2)$, and $(3,1)$.

13. The region under the graph of $\sqrt{x}$ on $[0,1]$ is revolved around the y axis. Sketch the resulting solid and find its volume. Relate the result to Example 5 of the previous section.
14. Find the volume in Example 4 by the slice method.
15. A cylindrical hole of radius $\frac{1}{2}$ is drilled through the center of a ball of radius 1. Use the shell method to find the volume of the resulting solid.

16. Find the volume in Exercise 15 by the slice method.
17. Find the volume of the solid torus obtained by rotating the disk $(x-3)^2+y^2 \leqslant 4$ about the y axis.
18. Find the volume of the solid torus obtained by rotating the disk $x^2+(y-5)^2 \leqslant 9$ about the x axis.
19. A spherical shell of radius r and thickness h is, by definition, the region between two concentric spheres of radius $r-h/2$ and $r+h/2$.
    (a) Find a formula for the volume $V(r,h)$ of a spherical shell of radius r and thickness h.
    (b) For fixed r, what is $(d/dh)V(r,h)$ when $h=0$? Interpret your result in terms of the surface area of the sphere.
20. In Exercise 19, find $(d/dr)V(r,h)$ when h is held fixed. Give a geometric interpretation of your answer.

★21. (a) Find the volume of the solid torus $T_{a,b}$ obtained by rotating the disk with radius a and center $(b,0)$ about the y axis, $0<a<b$.
    (b) What is the volume of the region between the solid tori $T_{a,b}$ and $T_{a+h,b}$, assuming $0<a+h<b$?
    (c) Using the result in (b), guess a formula for the area of the torus which is the surface of $T_{a,b}$. (Compare Exercise 19).

★22. Let $f(x)$ and $g(y)$ be inverse functions with $f(a)=\alpha$, $f(b)=\beta$, $0\leqslant a<b$, $0\leqslant \alpha<\beta$. Show that

$$2\pi\int_{\alpha}^{\beta} yg(y)\,dy = b\pi\beta^2 - a\pi\alpha^2 - \pi\int_a^b [f(x)]^2\,dx.$$

Interpret this statement geometrically.

★23. Use Exercise 22 to compute the volume of the solid obtained by revolving the graph $y=\cos^{-1}x$, $0\leqslant x\leqslant 1$, about the x axis.

9.3 Average Values and the Mean Value Theorem for Integrals

The average height of a region under a graph is its area divided by the length of the base.

The average value of a function on an interval will be defined in terms of an integral, just as the average or mean of a list $a_1,\ldots,a_n$ of n numbers is defined in terms of a sum as $(1/n)\sum_{i=1}^n a_i$.

If a grain dealer buys wheat from n farmers, buying b_i bushels from the ith farmer at the price of p_i dollars per bushel, the average price is determined not by taking the simple average of the p_i's, but rather by the "weighted average":

$$p_{\text{average}} = \frac{\sum_{i=1}^n p_i b_i}{\sum_{i=1}^n b_i} = \frac{\text{total dollars}}{\text{total bushels}}.$$

If a cyclist changes speed intermittently, travelling at v_1 miles per hour from t_0 to t_1, v_2 miles per hour from t_1 to t_2, and so on up to time t_n, then the average speed for the trip is

$$v_{\text{average}} = \frac{\sum_{i=1}^n v_i(t_i-t_{i-1})}{\sum_{i=1}^n (t_i-t_{i-1})} = \frac{\text{total miles}}{\text{total hours}}.$$

If, in either of the last two examples, the b_i's or (t_i-t_{i-1})'s are all equal, then the average value is simply the usual average of the p_i's or the v_i's.

If f is a step function on $[a,b]$ and we have a partition $(x_0,x_1,\ldots,x_n)$ with $f(x)=k_i$ on (x_{i-1},x_i), then the *average value* of f on the interval $[a,b]$ is defined to be

$$\overline{f(t)}_{[a,b]} = \frac{\sum_{i=1}^n k_i\Delta x_i}{\sum_{i=1}^n \Delta x_i}. \tag{1}$$

In other words, each interval is weighted by its length.

How can we define the average value of a function which is not a step function? For instance, it is common to talk of the average temperature at a

place on earth, although the temperature is not a step function. We may rewrite (1) as

$$\overline{f(x)}_{[a,b]} = \frac{\int_a^b f(x)\,dx}{b-a}, \tag{2}$$

and this leads us to adopt formula (2) as the definition of the average value for any integrable function f, not just a step function.

Average Value

If the function f has an integral on $[a,b]$, then the average value $\overline{f(x)}_{[a,b]}$ of f on $[a,b]$ is defined by the formula

$$\overline{f(x)}_{[a,b]} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example 1 Find the average value of $f(x) = x^2$ on $[0,2]$.

Solution By definition, we have

$$\overline{x^2}_{[0,2]} = \frac{1}{2-0}\int_0^2 x^2\,dx = \frac{1}{2}\cdot\frac{1}{3}x^3\Big|_0^2 = \frac{4}{3}. \ \blacktriangle$$

Example 2 Show that if $v = f(t)$ is the velocity of a moving object, then the definition of $\bar{v}_{[a,b]}$ agrees with the usual notion of average velocity.

Solution By the definition,

$$\bar{v}_{[a,b]} = \frac{1}{b-a}\int_a^b v\,dt;$$

but $\int_a^b v\,dt$ is the distance travelled between $t = a$ and $t = b$, so $\bar{v}_{[a,b]} =$ (distance travelled)/(time of travel), which is the usual definition of average velocity. $\blacktriangle$

Example 3 Find the average value of $\sqrt{1-x^2}$ on $[-1,1]$.

Solution By the formula for average values, $\overline{\sqrt{1-x^2}}_{[-1,1]} = (\int_{-1}^1\sqrt{1-x^2}\,dx)/2$; but $\int_{-1}^1\sqrt{1-x^2}\,dx$ is the area of the upper semicircle of $x^2+y^2=1$, which is $\frac{1}{2}\pi$, so $\overline{\sqrt{1-x^2}}_{[-1,1]} = \pi/4 \approx 0.785$. $\blacktriangle$

Example 4 Find $\overline{x^2\sin x^3}_{[0,\pi]}$.

Solution

$$\begin{aligned}\overline{x^2\sin(x^3)}_{[0,\pi]} &= \frac{1}{\pi}\int_0^\pi x^2\sin x^3\,dx \\ &= \frac{1}{\pi}\int_0^{\pi^3}\sin u\,\frac{du}{3} \qquad \text{(substituting } u = x^3\text{)} \\ &= \frac{1}{3\pi}(-\cos u)\Big|_0^{\pi^3} \\ &= \frac{1}{3\pi}(1-\cos\pi^3) \approx 0.0088. \ \blacktriangle\end{aligned}$$

We may rewrite the definition of the average value in the form

$$\int_a^b f(x)\,dx = \overline{f(x)}_{[a,b]}(b-a),$$

and the right-hand side can be interpreted as the integral of a constant function:

$$\int_a^b f(x)\,dx = \int_a^b \overline{f(x)}_{[a,b]}\,dx.$$

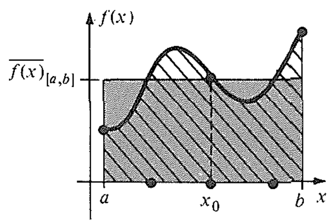


Figure 9.3.1. The average value is defined so that the area of the rectangle equals the area under the graph. The dots on the x axis indicate places where the average value is attained.

Geometrically, the average value is the height of the rectangle with base $[a,b]$ which has the same area as the region under the graph of f (see Fig. 9.3.1). Physically, if the graph of f is a picture of the surface of wavy water in a narrow channel, then the average value of f is the height of the water when it settles.

An important property of average values is given in the following statement:

If $m \leqslant f(x) \leqslant M$ for all x in $[a,b]$, then $m \leqslant \overline{f(x)}_{[a,b]} \leqslant M$.

Indeed, the integrals $\int_a^b m\,dx$ and $\int_a^b M\,dx$ are lower and upper sums for f on $[a,b]$, so

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a).$$

Dividing by $(b-a)$ gives the desired result.

By the extreme value theorem (Section 3.5), $f(x)$ attains a minimum value m and a maximum value M on $[a,b]$. Then $m \leqslant f(x) \leqslant M$ for x in $[a,b]$, so $\overline{f(x)}_{[a,b]}$ lies between m and M, by the preceding proposition. By the first version of the intermediate value theorem (Section 3.1), applied to the interval between the points where $f(x) = m$ and $f(x) = M$, we conclude that there is a x_0 in this interval (and thus in $[a,b]$), such that $f(x_0) = \overline{f(x)}_{[a,b]}$.

In other words, we have proved that the average value of a continuous function on an interval is always attained somewhere on the interval. This result is known as the *mean value theorem for integrals*.

Mean Value Theorem for Integrals

Let f be continuous on $[a,b]$. Then there is a point x_0 in (a,b) such that

$$f(x_0) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Notice that in Fig. 9.3.1, the mean value is attained at three different points.

Example 5 Give another proof of the mean value theorem for integrals by using the fundamental theorem of calculus and the mean value theorem for derivatives.

Solution Let f be continuous on $[a,b]$, and define $F(x) = \int_a^x f(s)\,ds$. By the fundamental theorem of calculus (alternative version), $F'(x) = f(x)$ for x in (a,b). (Exercise 29 asks you to verify that F is continuous at a and b—we accept it here.) By the mean value theorem for derivatives, there is some x_0 in (a,b) such that

$$F'(x_0) = \frac{F(b)-F(a)}{b-a}.$$

Substituting for F and F' in terms of f, we have

$$f(x_0) = \frac{\int_a^b f(x)\,dx - \int_a^a f(x)\,dx}{b-a} = \frac{\int_a^b f(x)\,dx}{b-a} = \overline{f(x)}_{[a,b]},$$

which establishes the mean value theorem for integrals. ▲

Exercises for Section 9.3

In Exercises 1–4, find the average value of the given function on the given interval.

1. x^3 on $[0,1]$
2. x^2+1 on $[1,2]$
3. $x/(x^2+1)$ on $[1,2]$
4. $\cos^2 x \sin x$ on $[0,\pi/2]$

Calculate each of the average values in Exercises 5–16.

5. $\overline{x^3}_{[0,2]}$
6. $\overline{z^3+z^2+1}_{[1,2]}$
7. $\overline{1/(1+t^2)}_{[-1,1]}$
8. $\overline{[(x^3+x-2)/(x^2+1)]}_{[-1,1]}$
9. $\overline{\sin^{-1}x}_{[0,1]}$
10. $\overline{\sin^{-1}x}_{[-1/2,0]}$
11. $\overline{\sin x\cos 2x}_{[0,\pi/2]}$
12. $\overline{(x^2+x-1)\sin x}_{[0,\pi/4]}$
13. $\overline{x^3+\sqrt{1/x}}_{[1,3]}$
14. $\overline{\sqrt{1-t^2}}_{[0,1]}$
15. $\overline{\sin^2 x}_{[0,\pi]}$
16. $\overline{\ln x}_{[1,e]}$
17. What was the average temperature in Goose Brow on June 13, 1857? (See Fig. 9.3.2).

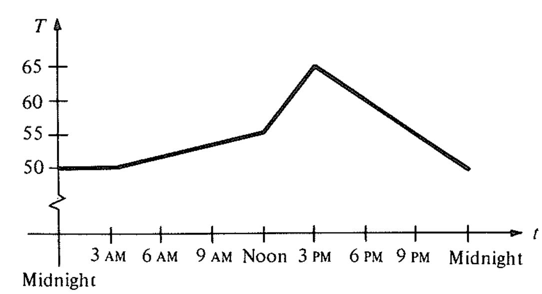


Figure 9.3.2. Temperature in Goose Brow on June 13, 1857.

18. Find the average temperature in Goose Brow (Fig. 9.3.2) during the periods midnight to 3P.M. and 3P.M. to midnight. How is the average over the whole day related to these numbers?
19. (a) Find $\overline{t^2+3t+2}_{[0,x]}$ as a function of x. (b) Evaluate this function of x for $x = 0.1, 0.01, 0.0001$. Try to explain what is happening.
20. Find $\overline{\cos\theta}_{[\pi,\pi+\theta]}$ as a function of θ and evaluate the limit as $\theta \to 0$.
21. Show that if $\overline{f'(x)}_{[a,b]} = 0$ then $f(b) = f(a)$.
22. Show that if $a < b < c$, then
$$\overline{f(t)}_{[a,c]} = \left(\frac{b-a}{c-a}\right)\overline{f(t)}_{[a,b]} + \left(\frac{c-b}{c-a}\right)\overline{f(t)}_{[b,c]}.$$

★23. How is the average of $f(x)$ on $[a,b]$ related to that of $f(x)+k$ for a constant k? Explain the answer in terms of a graph.

★24. If $f(x) = g(x) + h(x)$ on $[a,b]$, show that the average of f on $[a,b]$ is the sum of the averages of g and h on $[a,b]$.

★25. Suppose that f' exists and is continuous on $[a,b]$. Prove the mean value theorem for derivatives from the mean value theorem for integrals.

★26. Let f be defined on the real line and let $a(x) = \overline{f(x)}_{[0,x]}$.
(a) Derive the formula $a'(x) = (1/x)[f(x) - a(x)]$.
(b) Interpret the formula in the cases $f(x) = a(x)$, $f(x) < a(x)$, and $f(x) > a(x)$.
(c) When baseball players strike out, it lowers their batting average more at the beginning of the season than at the end. Explain why.

★27. The *geometric mean* of the positive numbers $a_1, \cdots, a_n$ is the nth root of the product $a_1 \cdots a_n$. Define the geometric mean of a positive function $f(x)$ on $[a,b]$. [*Hint:* Use logarithms.]

★28. (a) Use the idea of Exercise 27 to prove the arithmetic-geometric mean inequality (see Example 12, Section 3.5). [*Hint:* Use the fact that e^x is concave upwards.] (b) Generalize from numbers to functions.

★29. If f is continuous on $[a,b]$ and $F(x) = \int_a^x f(s)\,ds$, verify directly using the definition of continuity in Section 11.1 that F is continuous on $[a,b]$.

★30. (a) At what point of the interval $[0,a]$ is the average value of e^x achieved?
(b) Denote the expression found in part (a) by $p(a)$. Evaluate $p(a)$ for $a = 1, 10, 100, 1000$ and $a = 0.1, 0.01, 0.0001$, and 0.000001. Be sure that your answers are reasonable.
(c) Guess the limits $\lim_{a\to 0} p(a)/a$ and $\lim_{a\to\infty} p(a)/a$.

9.4 Center of Mass

The center of mass of a region is the point where it balances.

An important problem in mechanics, which was considered by Archimedes, is to locate the point on which a plate of some given irregular shape will balance (Fig. 9.4.1). This point is called the center of mass, or center of gravity, of the plate. The center of mass can also be defined for solid objects, and its applications range from theoretical physics to the problem of arranging wet towels to spin in a washing machine.

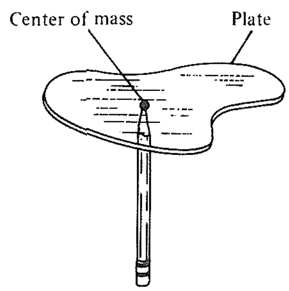


Figure 9.4.1. The plate balances when supported at its center of mass.

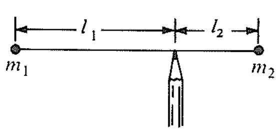


Figure 9.4.2. The support is at the center of mass when $m_1l_1 = m_2l_2$.

To give a mathematical definition of the center of mass, we begin with the ideal case of two point masses, m_1 and m_2, attached to a light rod whose mass we neglect. (Think of a see-saw.) If we support the rod (see Fig. 9.4.2) at a point which is at distance l_1 from m_1 and distance l_2 from m_2, we find that the rod tilts down at m_1 if $m_1l_1 > m_2l_2$ and down at m_2 if $m_1l_1 < m_2l_2$. It balances when

$$m_1l_1 = m_2l_2\,. \tag{1}$$

One can derive this balance condition from basic physical principles, or one may accept it as an experimental fact; we will not try to prove it here, but rather study its consequences.

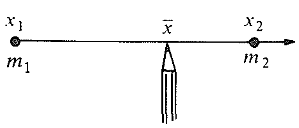


Figure 9.4.3. The center of mass is at $\bar{x}$ if $m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$.

Suppose that the rod lies along the x axis, with m_1 at x_1 and m_2 at x_2. Let $\bar{x}$ be the position of the center of mass. Comparing Figs. 9.4.2 and 9.4.3, we see that $l_1 = \bar{x} - x_1$ and $l_2 = x_2 - \bar{x}$, so formula (1) may be rewritten as $m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$. Solving for $\bar{x}$ gives the explicit formula

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}\,. \tag{2}$$

We may observe that the position of the center of mass is just the *weighted average* of the positions of the individual masses. This suggests the following generalization.

Center of Mass on the Line

If n masses, $m_1, m_2, \ldots, m_n$, are placed at the points $x_1, x_2, \ldots, x_n$, respectively, their center of mass is located at

$$\bar{x} = \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i}\,. \tag{3}$$

We may accept formula (3), as we did formula (1), as a physical fact, or we may derive it (see Example 1) from formula (2) and the following principle, which is also accepted as a general physical fact.

> **Consolidation Principle**
>
> If a body B is divided into two parts, B_1 and B_2, with masses M_1 and M_2, then the center of mass of the body B is located as if B consisted of two point masses: M_1 located at the center of mass of B_1, and M_2 located at the center of mass of B_2.

Example 1 Using formula (2) and the consolidation principle, derive formula (3) for the case of three masses.

Solution We consider the body B consisting of m_1, m_2, and m_3 as divided into B_1, consisting of m_1 and m_2, and B_2, consisting of m_3 alone. (See Fig. 9.4.4.)

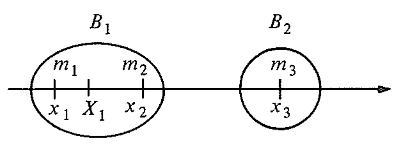


Figure 9.4.4. Center of mass of three points by the consolidation principle.

By formula (2) we know that X_1, the center of mass of B_1, is given by

$$X_1 = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}.$$

The mass M_1 of B_1 is $m_1 + m_2$. The body B_2 has center of mass at $X_2 = x_3$ and mass $M_2 = m_3$. Applying formula (2) once again to the point masses M_1 at X_1 and M_2 at X_2 gives the center of mass $\bar{x}$ of B by the consolidation principle:

$$\bar{x} = \frac{M_1X_1 + M_2X_2}{M_1 + M_2} = \frac{(m_1 + m_2)\left(\dfrac{m_1x_1 + m_2x_2}{m_1 + m_2}\right) + m_3x_3}{(m_1 + m_2) + m_3}$$
$$= \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3},$$

which is exactly formula (3) for $n = 3$. ▲

Example 2 Masses of 10, 20, and 25 grams are located at $x_1 = 0$, $x_2 = 5$, and $x_3 = 12$ centimeters, respectively. Locate the center of mass.

Solution Using formula (3), we have

$$\bar{x} = \frac{10(0) + 20(5) + 25(12)}{10 + 20 + 25} = \frac{400}{55} = \frac{80}{11} \approx 7.27 \text{ centimeters. ▲}$$

Now let us study masses in the plane. Suppose that the masses $m_1, m_2, \dots, m_n$ are located at the points $(x_1, y_1), \dots, (x_n, y_n)$. We imagine the masses as being attached to a weightless card, and we seek a point $(\bar{x}, \bar{y})$ on the card where it will balance. (See Fig. 9.4.5.)

To locate the center of mass $(\bar{x}, \bar{y})$, we note that a card which balances on the *point* $(\bar{x}, \bar{y})$ will certainly balance along any *line* through $(\bar{x}, \bar{y})$. Take, for instance, a line parallel to the y axis (Fig. 9.4.6). The balance along this line will not be affected if we move each mass parallel to the line so that m_1, m_2, m_3, and m_4 are lined up parallel to the x axis (Fig. 9.4.7).

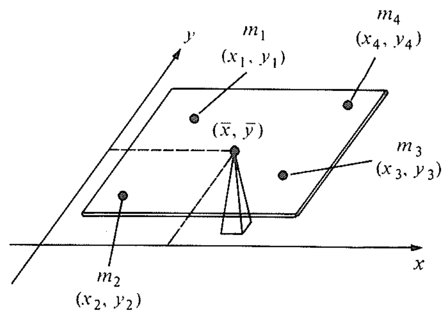


Figure 9.4.5. The card balances at the center of mass.

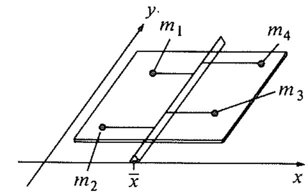


Figure 9.4.6. If the card balances at a point, it balances along any line through that point.

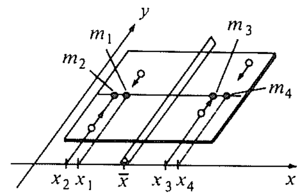


Figure 9.4.7. Moving the masses parallel to a line does not affect the balance along this line.

Now we can apply the balance equation (3) for masses in a line to conclude that the x component $\bar{x}$ of the center of mass is equal to the weighted average

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

of the x components of the point masses.

Repeating the construction for a balance line parallel to the x axis (we urge you to draw versions of Figs. 9.4.6 and 9.4.7 for this case), and applying formula (3) to the masses as lined up parallel to the y axis, we conclude that

$$\bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}.$$

These two equations completely determine the position of the center of mass.

Center of Mass in the Plane

If n masses, $m_1, m_2, \ldots, m_n$, are placed at the n points (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$, respectively, then their center of mass is located at $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \quad \text{and} \quad \bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}. \tag{4}$$

Example 3 Masses of 10, 15, and 30 grams are located at (0, 1), (1, 1), and (1, 0). Find their center of mass.

Solution Applying formula (4), with $m_1 = 10$, $m_2 = 15$, $m_3 = 30$, $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $y_1 = 1$, $y_2 = 1$, and $y_3 = 0$, we have

$$\bar{x} = \frac{10 \cdot 0 + 15 \cdot 1 + 30 \cdot 1}{10 + 15 + 30} = \frac{9}{11}$$

and

$$\bar{y} = \frac{10 \cdot 1 + 15 \cdot 1 + 30 \cdot 0}{10 + 15 + 30} = \frac{5}{11},$$

so the center of mass is located at $(\frac{9}{11}, \frac{5}{11})$. ▲

Example 4 Particles of mass 1, 2, 3, and 4 are located at successive vertices of a unit square. How far from the center of the square is the center of mass?

Solution We take the vertices of the square to be (0,0), (1,0), (1,1), and (0,1). (See Fig. 9.4.8.) The center is at $(\frac{1}{2}, \frac{1}{2})$ and the center of mass is located by formula (4):

$$\bar{x} = \frac{1 \cdot 0 + 2 \cdot 1 + 3 \cdot 1 + 4 \cdot 0}{1 + 2 + 3 + 4} = \frac{1}{2},$$

$$\bar{y} = \frac{1 \cdot 0 + 2 \cdot 0 + 3 \cdot 1 + 4 \cdot 1}{1 + 2 + 3 + 4} = \frac{7}{10}.$$

It is located $\frac{2}{10}$ unit above the center of the square. ▲

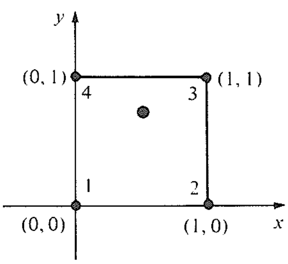


Figure 9.4.8. The center of mass of these four weighted points is located at $(\frac{1}{2}, \frac{7}{10})$.

We turn now from the study of center of mass for point masses to that for flat plates of various shapes.

A flat plate is said to be of *uniform density* if there is a constant ρ such that the mass of any piece of the plate is equal to ρ times the area of the piece. The number ρ is called the *density* of the plate. We represent a plate of uniform density by a region R in the plane; we will see that the value of ρ is unimportant as far as the center of mass is concerned.

A line l is called an *axis of symmetry* for the region R if the region R is taken into itself when the plane is flipped 180° around l (or, equivalently, reflected across l). For example, a square has four different axes of symmetry, a nonsquare rectangle two, and a circle infinitely many (see Fig. 9.4.9). Since a region will obviously balance along an axis of symmetry l, the center of mass must lie somewhere on l.

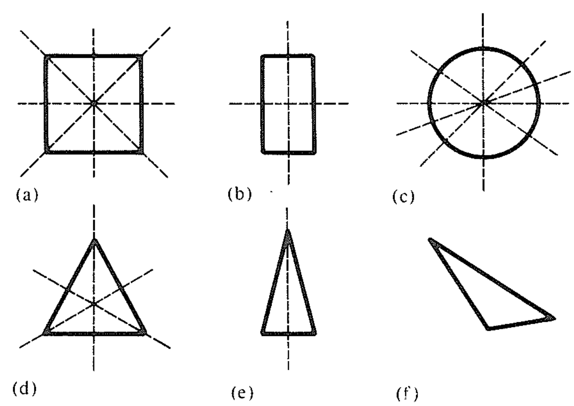


Figure 9.4.9. The axes of symmetry of various geometric figures.

Symmetry Principle

If l is an axis of symmetry for the plate R of uniform density, then the center of mass of R lies on l.

If a plate admits more than one axis of symmetry, then the center of mass must lie on all the axes. In this case, we can conclude that the center of mass lies at the point of intersection of the axes of symmetry. Looking at parts (a) through (d) of Fig. 9.4.9, we see that in each case the center of mass is located

at the "geometric center" of the figure. In case (e), we know only that the center of mass is on the altitude; in case (f), symmetry cannot be applied to determine the center of mass.

Using infinitesimals, we shall now derive formulas for the center of mass of the region under the graph of a function f, with uniform density ρ. As we did when computing areas, we think of the region under the graph of f on $[a, b]$ as being composed of "infinitely many rectangles of infinitesimal width." The rectangle at x with width dx has area $f(x)\,dx$ and mass $\rho f(x)\,dx$; its center of mass is located at $(x, \frac{1}{2}f(x))$ (by the symmetry principle) (see Fig. 9.4.10). [The center of mass is "really" at $(x + \frac{1}{2}dx, \frac{1}{2}f(x))$ but since the region is infinitesimally thin, we use $(x, \frac{1}{2}f(x))$—a more careful argument is given in the supplement to this section.]

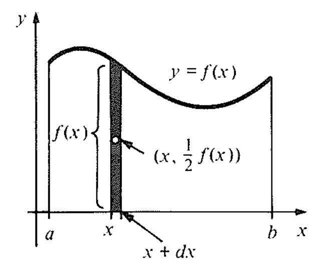


Figure 9.4.10. The "infinitesimal rectangle" has mass $\rho f(x)\,dx$ and center of mass at $(x, \frac{1}{2}f(x))$.

Now we apply the consolidation principle, but instead of summing, we replace the sums in formula (4) by integrals and arrive at the following result:

$$\bar{x} = \frac{\int_a^b x\rho f(x)\,dx}{\int_a^b \rho f(x)\,dx} = \frac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx}, \quad \text{and}$$

$$\bar{y} = \frac{\int_a^b \frac{1}{2}f(x)\rho f(x)\,dx}{\int_a^b \rho f(x)\,dx} = \frac{\frac{1}{2}\int_a^b [f(x)]^2\,dx}{\int_a^b f(x)\,dx}; \ (\rho \text{ cancels since it is constant}).$$

Since the center of mass depends only upon the region in the plane, and not upon the density ρ, we usually refer to $(\bar{x}, \bar{y})$ simply as the *center of mass of the region.*

Center of Mass of the Region under a Graph

The center of mass of a plate of uniform density represented by the region under the graph of a (non-negative) function $f(x)$ on $[a, b]$ is located at $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx} \quad \text{and} \quad \bar{y} = \frac{\frac{1}{2}\int_a^b [f(x)]^2\,dx}{\int_a^b f(x)\,dx}. \tag{5}$$

Example 5 Find the center of mass of the region under the graph of x^2 from 0 to 1.

Solution By formulas (5), with $f(x) = x^2$, $a = 0$, and $b = 1$,

$$\bar{x} = \frac{\int_0^1 x^3\,dx}{\int_0^1 x^2\,dx} = \frac{1/4}{1/3} = \frac{3}{4} \quad \text{and} \quad \bar{y} = \frac{\frac{1}{2}\int_0^1 x^4\,dx}{\int_0^1 x^2\,dx} = \frac{1/10}{1/3} = \frac{3}{10},$$

so the center of mass is located at $(\frac{3}{4}, \frac{3}{10})$. (See Fig. 9.4.11.) (You can verify

this result experimentally by cutting a figure out of stiff cardboard and seeing where it balances.) ▲

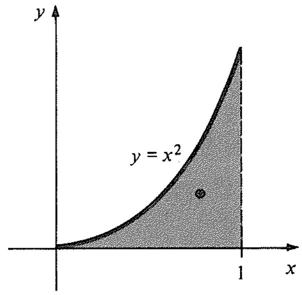


Figure 9.4.11. The center of mass of the shaded region is located at $(\frac{3}{4}, \frac{3}{10})$.

Example 6 Find the center of mass of a semicircular region of radius 1.

Solution We take the region under the graph of $\sqrt{1-x^2}$ on $[-1,1]$. Since the y axis is an axis of symmetry, the center of mass must lie on this axis; that is, $\bar{x}=0$. (You can also calculate $\int_{-1}^{1} x\sqrt{1-x^2}\,dx$ and find it to be zero.) By equation (5),

$$\bar{y}=\frac{\frac{1}{2}\int_{-1}^{1}(1-x^2)\,dx}{\int_{-1}^{1}\sqrt{1-x^2}\,dx}.$$

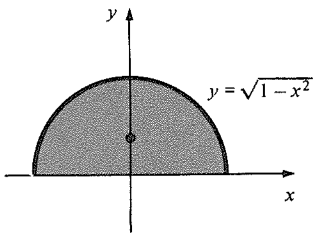


Figure 9.4.12. The center of mass of the semicircular region is located at $(0, 4/3\pi)$.

The denominator is the area $\pi/2$ of the semicircle. The numerator is $\frac{1}{2}\int_{-1}^{1}(1-x^2)\,dx=\frac{1}{2}[x-x^3/3]\big|_{-1}^{1}=\frac{2}{3}$, so

$$\bar{y}=\frac{2/3}{\pi/2}=\frac{4}{3\pi}\approx 0.42,$$

and so the center of mass is located at $(0, 4/3\pi)$ (see Fig. 9.4.12). ▲

Using the consolidation principle, we can calculate the center of mass of a region which is *not* under a graph by breaking it into simpler regions, as we did for areas in Section 4.

Example 7 Find the center of mass of the region consisting of a disk of radius 1 centered at the origin and the region under the graph of $\sin x$ on $(2\pi, 3\pi)$.

Solution The center of mass of the disk is at $(0,0)$, since the x and y axes are both axes of symmetry. For the region under the graph of $\sin x$ on $[2\pi, 3\pi]$, the line $x=\frac{5}{2}\pi$ is an axis of symmetry. To find the y coordinate of the center of mass, we use formula (5) and the identity $\sin^2 x=(1-\cos 2x)/2$ to obtain

$$\frac{\frac{1}{2}\int_{2\pi}^{3\pi}\sin^2 x\,dx}{\int_{2\pi}^{3\pi}\sin x\,dx}=\frac{\frac{1}{2}(x/2-\sin 2x/4)\big|_{2\pi}^{3\pi}}{-\cos x\big|_{2\pi}^{3\pi}}=\frac{(1/2)\cdot\pi/2}{2}=\frac{\pi}{8}\approx 0.393.$$

(Notice that this region is more "bottom heavy" than the semicircular region.)

By the consolidation principle, the center of mass of the total figure is the same as one consisting of two points: one at $(0,0)$ with mass $\rho\pi$ and one at $(\frac{5}{2}\pi, \pi/8)$ with mass 2ρ. The center of mass is, therefore, at $(\bar{x}, \bar{y})$, where

$$\bar{x}=\frac{\rho\pi\cdot 0+2\rho\cdot\frac{5}{2}\pi}{\rho\pi+\rho 2}=\frac{5\pi}{2+\pi}\quad\text{and}\quad\bar{y}=\frac{\rho\pi\cdot 0+2\rho\cdot\pi/8}{\rho\pi+\rho 2}=\frac{\pi/4}{2+\pi}.$$

Hence $(\bar{x}, \bar{y})$ is approximately (3.06, 0.15) (see Fig. 9.4.13). ▲

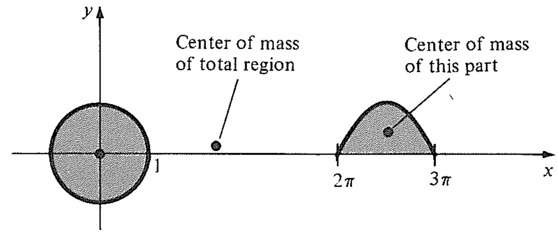


Figure 9.4.13. The center of mass is found by the consolidation principle.

Supplement to Section 9.4: A Derivation of the Center of Mass Formula (5) Using Step Functions

We begin by considering the case in which f is a step function on $[a,b]$ with $f(x) \geqslant 0$ for x in $[a,b]$. Let R be the region under the graph of f and let $(x_0, \dots, x_n)$ be a partition of $[a,b]$ such that f is a constant k_i on (x_{i-1}, x_i). Then R is composed of n rectangles $R_1, \dots, R_n$ of areas $k_i(x_i - x_{i-1}) = k_i \Delta x_i$ and masses $\rho k_i \Delta x_i = m_i$. By the symmetry principle, the center of mass of R_i is located at $(\bar{x}_i, \bar{y}_i)$, where $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$ and $\bar{y}_i = \frac{1}{2}k_i$. (See Fig. 9.4.14.)

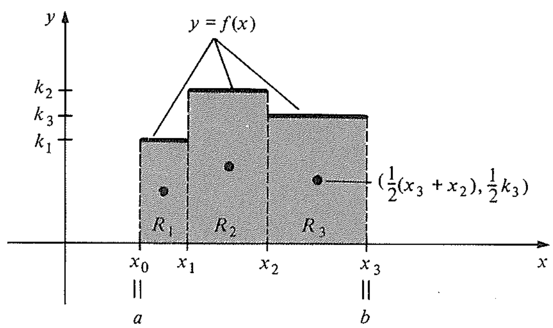


Figure 9.4.14. The center of mass of the shaded region is obtained by the consolidation principle.

Now we use the consolidation principle, extended to a decomposition into n pieces, to conclude that the center of mass of R is the same as the center of mass of masses $m_1, \dots, m_n$ placed at the points $(\bar{x}_1, \bar{y}_1), \dots, (\bar{x}_n, \bar{y}_n)$. By formula (4), we have, first of all,

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i \bar{x}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} \rho k_i \Delta x_i \left[\frac{1}{2}(x_{i-1} + x_i)\right]}{\sum_{i=1}^{n} \rho k_i \Delta x_i}.$$

We wish to rewrite the numerator and denominator as integrals, so that we can eventually treat the case where f is not a step function. The denominator is easy to handle. Factoring out ρ gives $\rho \sum_{i=1}^{n} k_i \Delta x_i$, which we recognize as $\rho \int_a^b f(x)\,dx$, the total mass of the plate. The numerator of $\bar{x}$ equals

$$\frac{1}{2}\rho \sum_{i=1}^{n} k_i(x_i - x_{i-1})(x_i + x_{i-1}) = \frac{1}{2}\rho \sum_{i=1}^{n} k_i\left(x_i^2 - x_{i-1}^2\right).$$

We notice that $k_i(x_i^2 - x_{i-1}^2) = \int_{x_{i-1}}^{x_i} 2k_i x\,dx$, which we can also write as $\int_{x_{i-1}}^{x_i} 2xf(x)\,dx$, since $f(x) = k_i$ on (x_{i-1}, x_i). Now the numerator of $\bar{x}$ becomes

$$\frac{1}{2}\rho \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} 2xf(x)\,dx = \frac{1}{2}\rho \int_a^b 2xf(x)\,dx = \rho \int_a^b xf(x)\,dx,$$

and we have

$$\bar{x} = \frac{\rho \int_a^b xf(x)\,dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx} \qquad \text{(cancelling } \rho\text{)}.$$

To find the y coordinate of the center of mass, we use the second half of formula (4):

$$\bar{y} = \frac{\sum_{i=1}^{n} m_i \bar{y}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} \rho k_i \Delta x_i (\frac{1}{2} k_i)}{\sum_{i=1}^{n} \rho k_i \Delta x_i}$$

The denominator is the total mass $\rho\int_a^b f(x)\,dx$, as before. The numerator is $\frac{1}{2}\rho\sum_{i=1}^{n} k_i^2 \Delta x_i$, and we recognize $\sum_{i=1}^{n} k_i^2 \Delta x_i$ as the integral $\int_a^b [f(x)]^2\,dx$ of the step function $[f(x)]^2$. Thus,

$$\bar{y} = \frac{\frac{1}{2}\rho\int_a^b [f(x)]^2\,dx}{\rho\int_a^b f(x)\,dx} = \frac{\frac{1}{2}\int_a^b [f(x)]^2\,dx}{\int_a^b f(x)\,dx}.$$

We have derived the formulas for $\bar{x}$ and $\bar{y}$ for the case in which $f(x)$ is a step function; however, they make sense as long as $f(x)$, $xf(x)$, and $[f(x)]^2$ are integrable on $[a, b]$. As usual, we carry over the same formula to general f, so formulas (5) are derived.

Exercises for Section 9.4

1. Redo Example 1 by choosing B_1 to consist of m_1 alone and B_2 to consist of m_2 and m_3.
2. Assuming formula (2) and the consolidation principle, derive formula (3) for the case of four masses by dividing the masses into two groups of two masses each.
3. Using formulas (2) and (3) for two and three masses, and the consolidation principle, derive formula (3) for four masses.
4. Assume that you have derived formula (3) from formula (2) and the consolidation principle for n masses. Now derive formula (3) for $n + 1$ masses.
5. Masses of 1, 3, 5, and 7 units are located at the points 7, 3, 5, and 1, respectively, on the x axis. Where is the center of mass?
6. Masses of 2, 4, 6, 8, and 10 units are located at the points $x_1 = 0$, $x_2 = 1$, $x_3 = 3$, $x_4 = -1$, and $x_5 = -2$ on the x axis. Locate the center of mass.
7. For each integer i from 1 to 100, a point of mass i is located at the point $x = i$. Where is the center of mass? (See Exercise 41(a), Section 4.1.)
8. Suppose that n equal masses are located at the points $1, 2, 3, \ldots, n$ on a line. Where is their center of mass?

In Exercises 9–12, find the center of mass for the given arrangement of masses.

9. 10 grams at $(1, 0)$ and 20 grams at $(1, 2)$.
10. 15 grams at $(-3, 2)$ and 30 grams at $(4, 2)$.
11. 5 grams at $(1, 1)$, 8 grams at $(3, 2)$, and 10 grams at $(0, 0)$.
12. 2 grams at $(4, 2)$, 3 grams at $(3, 2)$, and 4 grams at $(5, 3)$.
13. (a) Equal masses are placed at the vertices of an equilateral triangle whose base is the segment from $(0, 0)$ to $(1, 0)$. Where is the center of mass?
    (b) The mass at $(0, 0)$ is doubled. Where is the center of mass now?
14. Masses of 2, 3, 4, and 5 kilograms are placed at the points $(1, 2)$, $(1, 4)$, $(3, 5)$, and $(2, 6)$, respectively. Where should a mass of 1 kilogram be placed so that the configuration of five masses has its center of mass at the origin?
15. Verify the consolidation principle for the situation in which four masses in the plane are divided into two groups containing one mass and three masses each. (Assume that formula (3) holds for $n = 3$.)
16. Equal masses are placed at the points (x_1, y_1), (x_2, y_2), and (x_3, y_3). Show that their center of mass is at the intersection point of the medians of the triangle at whose vertices the masses are located.

Find the center of mass of the regions in Exercises 17–22.

17. The region under the graph of $4/x^2$ on $[1, 3]$.
18. The region under the graph of $1 + x^2 + x^4$ on $[-1, 1]$.
19. The region under the graph of $\sqrt{1 - x^2}$ on $[0, 1]$.
20. The region under the graph of $\sqrt{1 - x^2/a^2}$ on $[-a, a]$.
21. The triangle with vertices at $(0, 0)$, $(0, 2)$, and $(4, 0)$.
22. The triangle with vertices at $(1, 0)$, $(4, 0)$, and $(2, 3)$.
23. If, in formula (3), we have $a \leqslant x_i \leqslant b$ for all x_i, show that $a \leqslant \bar{x} \leqslant b$ as well. Interpret this statement geometrically.
24. Let a mass m_i be placed at position x_i on a line $(i = 1, \ldots, n)$. Show that the function $f(x) = \sum_{i=1}^{n} m_i (x - x_i)^2$ is minimized when x is the center of mass of the n particles.

25. Suppose that masses m_i are located at points x_i on the line and are moving with velocity $v_i = dx_i/dt$ $(i = 1, \ldots, n)$. The *total momentum* of the particles is defined to be $P = m_1v_1 + m_2v_2 + \cdots + m_nv_n$. Show that $P = Mv$, where M is the total mass and v is the velocity of the center of mass (i.e., the rate of change of the position of the center of mass with respect to time).
26. A mass m_i is at position $x_i = f_i(t)$ at time t. Show that if the force on m_i is $F_i(t)$, and $F_1(t) + F_2(t) = 0$, then the center of mass of m_1 and m_2 moves with constant velocity.
27. From a disk of radius 5, a circular hole with radius 2 and center 1 unit from the center of the disk is cut out. Sketch and find the center of mass of the resulting figure.
28. Suppose that $f(x) \leqslant g(x)$ for all x in $[a, b]$. Show that the center of mass of the region between the graphs of f and g on $[a, b]$ is located at $(\bar{x}, \bar{y})$, where
$$\bar{x} = \frac{\int_a^b x[g(x) - f(x)]\,dx}{\int_a^b [g(x) - f(x)]\,dx}, \quad \text{and}$$
$$\bar{y} = \frac{\frac{1}{2}\int_a^b [g(x) + f(x)][g(x) - f(x)]\,dx}{\int_a^b [g(x) - f(x)]\,dx}.$$
29. Find the center of mass of the region between the graphs of $\sin x$ and $\cos x$ on $[0, \pi/4]$. [*Hint:* Find the center of mass of each infinitesimal strip making up the region, or use Exercise 28.]
30. Find the center of mass of the region between the graphs of $-x^4$ and x^2 on $[-1, 1]$. (See the hint in Exercise 29.)
★31. Find the center of mass of the triangular region with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3). (For convenience, you may assume that $x_1 \leqslant x_2 \leqslant x_3$, $y_1 \leqslant y_3$, and $y_2 \leqslant y_3$.) Compare with Exercise 16.

9.5 Energy, Power, and Work

Energy is the integral of power over time, and work is the integral of force over distance.

Energy appears in various forms and can often be converted from one form into another. For instance, a solar cell converts the energy in light into electrical energy; a fusion reactor, in changing atomic structures, transforms nuclear energy into heat energy. Despite the variety of forms in which energy may appear, there is a common unit of measure for all these forms. In the MKS (meter-kilogram-second) system, it is the *joule*, which equals 1 kilogram meter^2 per second^2.

Energy is an "extensive" quantity. This means the following: the longer a generator runs, the more electrical energy it produces; the longer a light bulb burns, the more energy it consumes. The rate (with respect to time) at which some form of energy is produced or consumed is called the *power* output or input of the energy conversion device. Power is an *instantaneous* or "intensive" quantity. By the fundamental theorem of calculus, we can compute the total energy transformed between times a and b by integrating the power from a to b.

Power and Energy

Power is the rate of change of energy with respect to time:
$$P = \frac{dE}{dt}.$$
The total energy over a time period is the integral of power with respect to time:
$$E = \int_a^b P\,dt.$$

A common unit of measurement for power is the *watt*, which equals 1 joule per second. One *horsepower* is equal to 746 watts. The *kilowatt-hour* is a unit of energy equal to the energy obtained by using 1000 watts for 1 hour (3600 seconds)—that is, 3,600,000 joules.

Example 1 The power output (in watts) of a 60-cycle generator varies with time (measured in seconds) according to the formula $P = P_0\sin^2(120\pi t)$, where P_0 is the maximum power output. (a) What is the total energy output during an hour? (b) What is the average power output during an hour?

Solution (a) The energy output, in joules, is

$$E = \int_0^{3600} P_0 \sin^2(120\pi t)\, dt.$$

Using the formula $\sin^2\theta = (1 - \cos 2\theta)/2$, we find

$$E = \frac{1}{2}P_0\int_0^{3600}(1 - \cos 240\pi t)\, dt = \frac{1}{2}P_0\left[t - \frac{1}{240\pi}\sin 240\pi t\right]\Big|_0^{3600}$$

$$= \frac{1}{2}P_0\left[3600 - 0 - (0 - 0)\right] = 1800P_0.$$

(b) The average power output is the energy output divided by the time (see Section 9.3), or $1800\, P_0/3600 = \frac{1}{2}P_0$; in this case, half the maximum power output. ▲

A common form of energy is mechanical energy—the energy stored in the movement of a massive object (*kinetic energy*) or the energy stored in an object by virtue of its position (*potential energy*). The latter is illustrated by the energy we can extract from water stored above a hydroelectric power plant.

We accept the following principles from physics:

1. The kinetic energy of a mass m moving with velocity v is $\frac{1}{2}mv^2$.
2. The (gravitational) potential energy of a mass m at a height h is mgh (here g is the gravitational acceleration; $g = 9.8$ meters/(second)^2 = 32 feet/(second)^2.

The total force on a moving object is equal to the product of the mass m and the acceleration $dv/dt = d^2x/dt^2$. The unit of force is the *newton* which is 1 kilogram meter per second^2. If the force depends upon the position of the object, we may calculate the variation of the kinetic energy $K = \frac{1}{2}mv^2$ with position. We have

$$\frac{dK}{dx} = \frac{dK/dt}{dx/dt} = \frac{(d/dt)(\frac{1}{2}mv^2)}{v} = \frac{mv\, dv/dt}{v} = m\frac{dv}{dt} = F.$$

Applying the fundamental theorem of calculus, we find that the change ΔK of kinetic energy as the particle moves from a to b is $\int_a^b F\, dx$. Often we can divide the total force on an object into parts arising from identifiable sources (gravity, friction, fluid pressure). We are led to define the *work* W done by a particular force F on a moving object (even if there are other forces present) as $W = \int_a^b F\, dx$. Note that if the force F is constant, then the work done is simply the product of F with the *displacement* $\Delta x = b - a$. Accordingly, 1 joule equals 1 newton-meter.

Force and Work

The work done by a force on a moving object is the integral of the force with respect to position:

$$W = \int_a^b F\,dx.$$

If the force is constant,

Work = Force $\times$ Displacement.

If the total force F is a sum $F_1 + \cdots + F_n$, then we have

$$\Delta K = \int_a^b (F_1 + \cdots + F_n)\,dx = \int_a^b F_1\,dx + \cdots + \int_a^b F_n\,dx.$$

Thus the total change in kinetic energy is equal to the sum of the works done by the individual forces.

Example 2 The acceleration of gravity near the earth is $g = 9.8$ meters/(second)^2. How much work does a weight-lifter do in raising a 50-kilogram barbell to a height of 2 meters? (See Figure 9.5.1.)

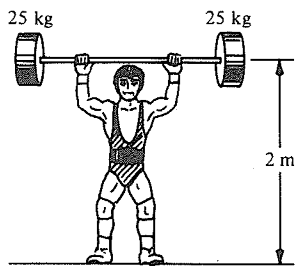


Figure 9.5.1. How much work did the weight-lifter do?

Solution We let x denote the height of the barbell above the ground. Before and after the lifts, the barbell is stationary, so the net change in kinetic energy is zero. The work done by the weight-lifter must be the *negative* of the work done by gravity. Since the pull of gravity is downward, its force is -9.8 meters per second^2 $\times$ 50 kilograms $= -490$ kilograms $\cdot$ meters per second^2 $= -490$ newtons; $\Delta x = 2$ meters, so the work done by gravity is -980 kilograms $\cdot$ meters^2 per second^2 $= -980$ joules. Thus the work done by the weight-lifter is 980 joules. (If the lift takes s seconds, the average power output is $(980/s)$ watts.) ▲

Example 3 Show that the *power* exerted by a force F on a moving object is Fv, where v is the velocity of the object.

Solution Let E be the energy content. By our formula for work, we have $\Delta E = \int_a^b F\,dx$, so $dE/dx = F$. To compute the power, which is the *time* derivative of E, we use the chain rule:

$$P = \frac{dE}{dt} = \frac{dE}{dx} \cdot \frac{dx}{dt} = Fv.$$

(In pushing a child on a swing, this suggests it is most effective to exert your force at the bottom of the swing, when the velocity is greatest. Are there any complicating factors?) ▲

Example 4 A pump is to empty the conical tank of water shown in Fig. 9.5.2. How much energy (in joules) is required for the job? (A cubic meter of water has mass 10^3 kilograms.)

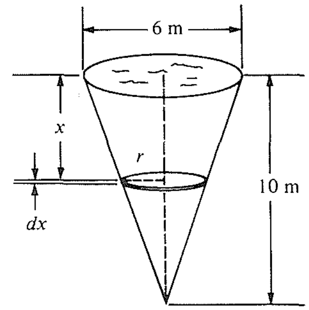


Figure 9.5.2. To calculate the energy needed to empty the tank, we add up the energy needed to remove slabs of thickness dx.

Solution Consider a layer of thickness dx at depth x, as shown in Fig. 9.5.2. By similar triangles, the radius is $r = \frac{3}{10}(10-x)$, so the volume of the layer is given by $\pi \cdot \frac{9}{100}(10-x)^2\,dx$ and its mass is $10^3 \cdot \pi \cdot \frac{9}{100}(10-x)^2\,dx = 90\pi(10-x)^2\,dx$. To lift this layer x meters to the top of the tank takes $90\pi(10-x)^2\,dx \cdot g \cdot x$ joules of work, where $g = 9.8$ meters per second^2 is the acceleration due to gravity (see Example 2). Thus, the total work done in emptying the tank is

$$\begin{aligned} 90g\pi\int_0^{10}(10-x)^2 x\,dx &= 90g\pi\left[100\frac{x^2}{2} - 20\frac{x^3}{3} + \frac{x^4}{4}\right]\Bigg|_0^{10} \\ &= 90g\pi(10^4)\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] \\ &= (90)(9.8)(\pi)(10^4)\left(\frac{1}{12}\right) \\ &\approx 2.3 \times 10^6 \text{ joules. } \blacktriangle \end{aligned}$$

Example 5 The pump which is emptying the conical tank in Example 4 has a power output of 10^5 joules per hour (i.e., 27.77 watts). What is the water level at the end of 6 minutes of pumping? How fast is the water level dropping at this time?

Solution The total energy required to pump out the top h meters of water is

$$\begin{aligned} 90g\pi\int_0^{h}(10-x)^2 x\,dx &= 90g\pi\left(100\frac{h^2}{2} - 20\frac{h^3}{3} + \frac{h^4}{4}\right) \\ &\approx 2770h^2\left(50 - \frac{20}{3}h + \frac{h^2}{4}\right). \end{aligned}$$

At the end of 6 minutes ($\frac{1}{10}$ hour), the pump has produced 10^4 joules of energy, so the water level is h meters from the top, where h is the solution of

$$2770h^2\left(50 - \frac{20}{3}h + \frac{h^2}{4}\right) = 10^4.$$

Solving this numerically by the method of bisection (see Section 3.1) gives $h \approx 0.27$ meter.

At the end of t hours, the total energy output is $10^5 t$ joules, so

$$90g\pi\int_0^{h}(10-x)^2 x\,dx = 10^5 t,$$

where h is the amount pumped out at time t. Differentiating both sides with

respect to t gives

$$90g\pi(10-h)^2h\frac{dh}{dt}=10^5, \quad \text{so} \quad \frac{dh}{dt}=\frac{10^4}{9g\pi(10-h)^2h}.$$

when $h=0.27$, this is 1.41 meters per hour. ▲

Supplement to Section 9.5: Integrating Sunshine

We will now apply the theory and practice of integration to compute the total amount of sunshine received during a day, as a function of latitude and time of year. If we have a horizontal square meter of surface, then the rate at which solar energy is received by this surface—that is, the *intensity* of the solar radiation—is proportional to the sine of the angle A of elevation of the sun above the horizon.[6] Thus the intensity is highest when the sun is directly overhead ($A=\pi/2$) and reduces to zero at sunrise and sunset.

The total energy received on day T must therefore be the product of a constant (which can be determined only by experiment, and which we will ignore) and the integral $E=\int_{t_0(T)}^{t_1(T)}\sin A\,dt$, where t is the time of day (measured in hours from noon) and $t_0(T)$ and $t_1(T)$ are the times of sunrise and sunset on day T. (When the sun is below the horizon, although $\sin A$ is negative, the solar intensity is simply zero.)

We presented a formula for $\sin A$ (formula (1) in the Supplement to Chapter 5, to be derived in the Supplement to Chapter 14), and used it to determine the time of sunset (formula (3) in the Supplement to Chapter 5). The time of sunrise is the negative of the time of sunset, so we have[7]

$$E=\int_{-S}^{S}\sin A\,dt, \tag{1}$$

where

$$\sin A=\cos l\sqrt{1-\sin^2\alpha\cos^2\left(\frac{2\pi T}{365}\right)}\cos\left(\frac{2\pi t}{24}\right)+\sin l\sin\alpha\cos\left(\frac{2\pi T}{365}\right)$$

and

$$S=\frac{24}{2\pi}\cos^{-1}\left[-\tan l\frac{\sin\alpha\cos(2\pi T/365)}{\sqrt{1-\sin^2\alpha\cos^2(2\pi T/365)}}\right]. \tag{2}$$

Here $\alpha\approx 23.5°$ is the inclination of the earth's axis from the perpendicular to the plane of the earth's orbit; l is the latitude of the point where the sunshine is being measured.

The integration will be simpler than you may expect. First of all, we simplify notation by writing k for the expression $\sin\alpha\cos(2\pi T/365)$, which appears so often. Then we have

$$\begin{aligned}E&=\int_{-S}^{S}\left[\cos l\sqrt{1-k^2}\cos\left(\frac{2\pi t}{24}\right)+(\sin l)k\right]dt\\&=\cos l\sqrt{1-k^2}\int_{-S}^{S}\cos\left(\frac{2\pi t}{24}\right)dt+(\sin l)k\int_{-S}^{S}dt.\end{aligned}$$

[6] We will justify this assertion in the Supplement to Chapter 14. We also note that, strictly speaking, it applies only if we neglect absorption by the atmosphere or assume that our surface is at the top of the atmosphere.

[7] All these calculations assume that there is a sunrise and sunset. In the polar regions during the summer, the calculations must be altered (see Exercise 5 below).

Integration gives

$$E = \cos l\sqrt{1-k^2}\left(\frac{24}{2\pi}\sin\frac{2\pi t}{24}\Big|_{-S}^{S}\right) + 2Sk\sin l$$

$$= \cos l\sqrt{1-k^2}\left(\frac{24}{2\pi}\right)\left(\sin\frac{2\pi S}{24} - \frac{\sin 2\pi(-S)}{24}\right) + 2Sk\sin l$$

$$= \frac{24}{\pi}\cos l\sqrt{1-k^2}\sin\frac{2\pi S}{24} + 2Sk\sin l.$$

The expression $\sin(2\pi S/24)$ can be simplified. Using the formula

$$\cos(2\pi S/24) = -(\tan l)\left(k/\sqrt{1-k^2}\right),$$

we get

$$\sin\frac{2\pi S}{24} = \sqrt{1-\cos^2\frac{2\pi S}{24}} = \sqrt{1-\frac{(\tan^2 l)k^2}{1-k^2}}$$

$$= \sqrt{\frac{1-k^2-(\tan^2 l)k^2}{1-k^2}}$$

$$= \sqrt{\frac{1-(1+\tan^2 l)k^2}{1-k^2}} = \sqrt{\frac{1-(\sec^2 l)k^2}{1-k^2}},$$

and so, finally, we get,

$$E = \frac{24}{\pi}\cos l\sqrt{1-(\sec^2 l)k^2} + \frac{24}{\pi}k\sin l\cos^{-1}\left[-\frac{(\tan l)k}{\sqrt{1-k^2}}\right].$$

Since both k and $\sqrt{1-k^2}$ appear, we can do even better by writing $k = \sin D$ (the number D is important in astronomy—it is called the *declination*), and we get

$$E = \frac{24}{\pi}\left[\cos l\sqrt{1-\sec^2 l\sin^2 D} + \sin l\sin D\cos^{-1}(-\tan l\tan D)\right].$$

Since we have already ignored a constant factor in E, we will also ignore the factor $24/\pi$. Incorporating $\cos l$ into the square root, we obtain as our final result

$$E = \sqrt{\cos^2 l - \sin^2 D} + \sin l\sin D\cos^{-1}(-\tan l\tan D), \tag{3}$$

where $\sin D = \sin\alpha\cos(2\pi T/365)$.

Plotting E as a function of l for various values of T leads to graphs like those in Figs. 3.5.4 and 9.5.3.

Example When does the equator receive the most solar energy? The least?

Solution At the equator, $l = 0$, so we have

$$E = \sqrt{1-\sin^2 D} = \sqrt{1-\sin^2\alpha\cos^2\left(\frac{2\pi T}{365}\right)}.$$

We see by inspection that E is largest when $\cos^2(2\pi T/365) = 0$—that is, when $T/365 = \frac{1}{4}$ or $\frac{3}{4}$; that is, on the first days of fall and spring: on these days $E = 1$. We note that E is smallest on the first days of summer and winter, when $\cos^2(2\pi/365) = 1$ and we have $E = \sqrt{1-\sin^2\alpha} = \cos\alpha = \cos 23.5° = 0.917$, or about 92% of the maximum value. ▲

Using this example we can standardize units in which E can be measured. One unit of E is the total energy received on a square meter at the equator on the first day of spring. All other energies may be expressed in terms of this unit.

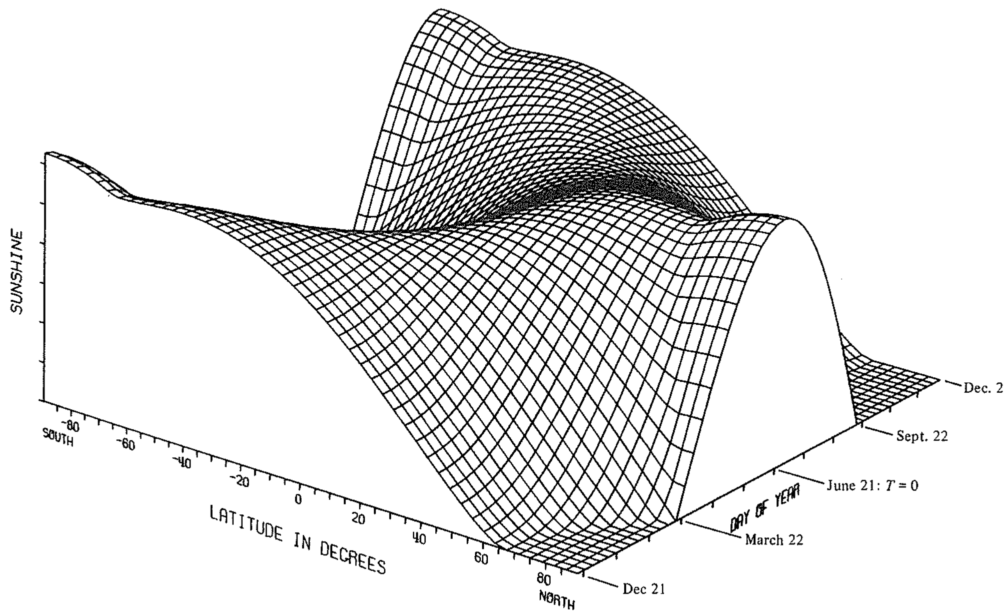


Figure 9.5.3. Computer-generated graph of the daily sunshine intensity on the earth as a function of day of the year and latitude.

Exercises for the Supplement to Section 9.5

1. Compare the solar energy received on June 21 at the Arctic Circle ($l = 90° - \alpha$) with that received at the equator.
2. What would the inclination of the earth need to be in order for E on June 21 to have the same value at the equator as at the latitude $90° - \alpha$?
3. (a) Express the total solar energy received over a whole year at latitude l by using summation notation. (b) Write down an integral which is approximately equal to this sum. Can you evaluate it?
4. Simplify the integral in the solution of Exercise 3(b) for the cases $l = 0$ (equator) and $l = 90° - \alpha$ (Arctic Circle). In each case, one of the two terms in the integrand can be integrated explicitly: find the integral of this term.
5. Find the total solar energy received at a latitude in the polar region on a day on which the sun never sets.
6. How do you think the climate of the earth would be affected if the inclination α were to become: (a) 10°? (b) 40°? (In each case, discuss whether the North Pole receives more or less energy during the year than the equator—see Exercise 5.)
7. Consider equation (3) for E. For $D = \pi/8$, compute dE/dl at $l = \pi/4$. Is your answer consistent with the graph in Fig. 9.5.3 (look in the plane of constant T)?
8. Determine whether a square meter at the equator or at the North Pole receives more solar energy: (a) during the month of February, (b) during the month of April, (c) during the entire year.

Exercises for Section 9.5

1. The power output (in watts) of a 60-cycle generator is $P = 1050\sin^2(120\pi t)$, where t is measured in seconds. What is the total energy output in an hour?
2. A worker, gradually becoming tired, has a power output of $30e^{-2t}$ watts for $0 \leqslant t \leqslant 360$, where t is the time in seconds from the start of a job. How much energy is expended during the job?
3. An electric motor is operating with power $15 + 2\sin(t\pi/24)$ watts, where t is the time in *hours* measured from midnight. How much energy is consumed in one day's operation?
4. The power output of a solar cell is $25\sin(\pi t/12)$ watts, where t is the time in *hours* after 6 A.M. How many joules of energy are produced between 6 A.M. and 6 P.M?

In Exercises 5–8, compute the work done by the given force acting over the given interval.

5. $F = 3x$; $0 \leqslant x \leqslant 1$.
6. $F = k/x^2$; $1 \leqslant x \leqslant 6$ (k a constant).
7. $F = 1/(4 + x^2)$; $0 \leqslant x \leqslant 1$.
8. $F = \sin^3 x\cos^2 x$; $0 \leqslant x \leqslant 2$. [*Hint:* Write $\sin^3 x = \sin x(1 - \cos^2 x)$.]
9. How much power must be applied to raise an object of mass 1000 grams at a rate of 10 meters per second (at the Earth's surface)?
10. The gravitational force on an object at a distance r from the center of the earth is k/r^2, where k is a constant. How much work is required to move the object:
    (a) From $r = 1$ to $r = 10$?
    (b) From $r = 1$ to $r = 1000$?
    (c) From $r = 1$ to $r = 10{,}000$?
    (d) From $r = 1$ to "$r = \infty$"?
11. A particle with mass 1000 grams has position $x = 3t^2 + 4$ meters at time t seconds. (a) What is the kinetic energy at time t? (b) What is the rate at which power is being supplied to the object at time $t = 10$?
12. A particle of mass 20 grams is at rest at $t = 0$, and power is applied at the rate of 10 joules per second. (a) What is the energy at time t? (b) If all the energy is kinetic energy, what is the velocity at time t? (c) How far has the particle moved at the end of t seconds? (d) What is the force on the particle at time t?
13. A force $F(x) = -3x$ newtons acts on a particle between positions $x = 1$ and $x = 0$. What is the change in kinetic energy of the particle between these positions?
14. A force $F(x) = 3x\sin(\pi x/2)$ newtons acts on a particle between positions $x = 0$ and $x = 2$. What is the increase in kinetic energy of the particle between these positions?
15. (a) The power output of an electric generator is $25\cos^2(120\pi t)$ joules per second. How much energy is produced in 1 hour? (b) The output of the generator in part (a) is converted, with 80% efficiency, into the horizontal motion of a 250-gram object. How fast is the object moving at the end of 1 minute?
16. The generator in Exercise 15 is used to lift a 500-kilogram weight and the energy is converted via pulleys with 75% efficiency. How high can it lift the weight in an hour?

Exercises 17–20 refer to Figure 9.5.4.

17. How much energy is required to pump all the water out of the swimming pool?
18. Suppose that a mass equal to that of the water in the pool were moving with kinetic energy equal to the result of Exercise 17. What would its velocity be?
19. Repeat Exercise 17 assuming that the pool is filled with a liquid three times as dense as water.
20. Repeat Exercise 18 assuming that the pool is filled with a liquid three times as dense as water.

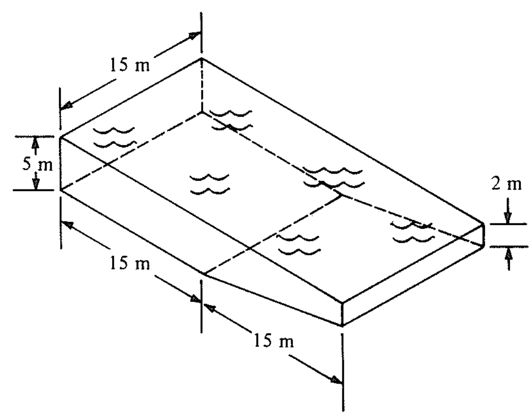


Figure 9.5.4. How much energy is required to empty this pool of water?

21. Suppose that a spring has a natural length of 10 centimeters, and that a force of 3 newtons is required to stretch it to 15 centimeters. How much work is needed to compress the spring to 5 centimeters?
22. If all the energy in the compressed spring in Exercise 21 is used to fire a ball with a mass of 20 grams, how fast will the ball travel?
23. How much work is required to fill the tank in Figure 9.5.5 with water from ground level?
24. A solid concrete monument is built in the pyramid shape of Fig. 9.5.6. Assume that the concrete weighs 260 pounds per cubic foot. How much work is done in erecting the monument? ($g = 32$ feet per second^2; express your answer in units of pound-feet.)

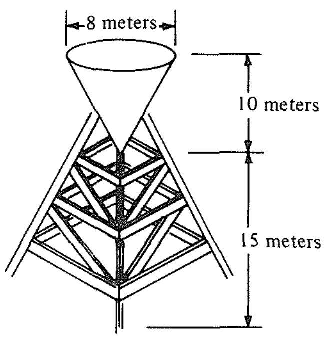


Figure 9.5.5. How much energy is needed to fill this tank?

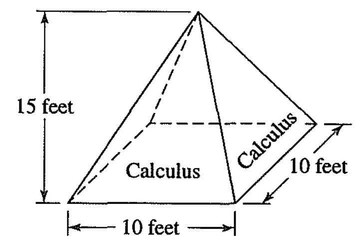


Figure 9.5.6. How much energy is needed to erect this monument?

Review Exercises for Chapter 9

In Exercises 1–4, find the volume of the solid obtained by rotating the region under the given graph about (a) the x axis and (b) the y axis.

1. $y = \sin x$, $0 \leqslant x \leqslant \pi$
2. $y = 3\sin 2x$, $0 \leqslant x \leqslant \pi/4$
3. $y = e^x$, $0 \leqslant x \leqslant \ln 2$
4. $y = 5e^{2x}$, $0 \leqslant x \leqslant \ln 4$
5. A cylindrical hole of radius $\frac{1}{3}$ is drilled through the center of a ball of radius 1. What is the volume of the resulting solid?
6. A wedge is cut in a tree of radius 1 meter by making two cuts to the center, one horizontally, and one at an angle of 20° to the first. Find the volume of the wedge.
7. Find the volume of the "football" whose dimensions are shown in Fig. 9.R.1. The two arcs in the figure are segments of parabolas.

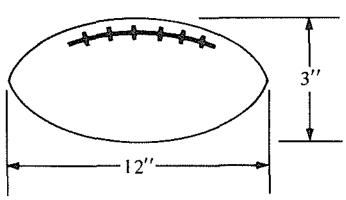


Figure 9.R.1. Find the volume of the football.

8. Imagine the "football" in Fig. 9.R.1, formed by revolving a parabola, to be solid. A hole with radius 1 inch is drilled along the axis of symmetry. How much material is removed?

In Exercises 9–12, find the average value of each function on the stated interval.

9. $1 + t^3$, $0 \leqslant t \leqslant 1$
10. $t\sin(t^2)$, $\pi \leqslant t \leqslant 3\pi/2$
11. xe^x, $0 \leqslant x \leqslant 1$
12. $\dfrac{1}{1+x^2}$, $1 \leqslant x \leqslant 3$
13. If $\int_0^2 f(x)\,dx = 4$, what is the average value of $g(x) = 3f(x)$ on $[0,2]$?
14. If $f(x) = kg(cx)$ on $[a,b]$, how is the average of f on $[a,b]$ related to that of g on $[ac,bc]$?
15. Show that for some x in $[0,\pi]$, $\dfrac{\pi}{2+\cos x}$ is equal to $\displaystyle\int_0^\pi \frac{d\theta}{2+\cos\theta}$.
16. (a) Prove that
$$1/\sqrt{2} \leqslant \int_0^1 \left[dt/\sqrt{t^3+1} \right] \leqslant 1.$$
(b) Prove that $\int_0^1 [dt/\sqrt{t^3+1}] = \sin\theta$ for some θ, $\pi/4 \leqslant \theta \leqslant \pi/2$.

In Exercises 17–22, let μ be the average value of f on $[a,b]$. Then the average value of $[f(x) - \mu]^2$ on $[a,b]$ is called the *variance* of f on $[a,b]$, and the square root of the variance is called the *standard deviation* of f on $[a,b]$ and is denoted σ. Find the average value, variance, and standard deviation of each of the following functions on the interval specified.

17. x^2 on $[0,1]$
18. $3 + x^2$ on $[0,1]$
19. xe^x on $[0,1]$
20. $\sin 2x$ on $[0,4\pi]$
21. $f(x) = 1$ on $[0,1]$ and 2 on $(1,2]$.
22. $f(x) = \begin{cases} 2 & \text{on } [0,1] \\ 3 & \text{on } (1,2] \\ 1 & \text{on } (2,3] \\ 5 & \text{on } (3,4] \end{cases}$
23. Let the region under the graph of a positive function $f(x)$, $a \leqslant x \leqslant b$, be revolved about the x axis to form a solid S. Suppose this solid has a mass density of $\rho(x)$ grams per cubic centimeter at a distance x along the x axis. (a) Find a formula for the mass of S. (b) If $f(x) = x^2$, $a = 0$ and $b = 1$, and $\rho(x) = (1 + x^4)$, find the mass of S.
24. A rod has linear mass density $\mu(x)$ grams per centimeter at the point x along its length. If the rod extends from $x = a$ to $x = b$, find a formula for the location of the rod's center of mass.

In Exercises 25–28, find the center of mass of the region under the given graph on the given interval.

25. $y = x^4$ on $[0,2]$
26. $y = x^3 + 2$ on $[0,1]$
27. $y = \ln(1+x)$ on $[0,1]$
28. $y = e^x$ on $[1,2]$

29. Find the center of mass of the region between the graphs of $y = x^3$ and $y = -x^2$ between $x = 0$ and $x = 1$ (see Exercise 28, Section 9.4).
30. Find the center of mass of the region composed of the region under the graph $y = \sin x$, $0 \leqslant x \leqslant \pi$, and the circle with center at $(5, 0)$ and radius 1.
31. Over a time period $0 \leqslant t \leqslant 6$ (t measured in minutes), an engine is consuming power at a rate of $20 + 5te^{-t}$ watts. What is (a) the total energy consumed? (b) The average power used?
32. Water is being pumped from a deep, irregularly shaped well at a constant rate of $3\frac{1}{2}$ cubic meters per hour. At a certain instant, it is observed that the water level is dropping at a rate of 1.2 meters per hour. What is the cross-sectional area of the well at that depth?
33. A force $F(x) = 30\sin(\pi x/4)$ newtons acts on a particle between positions $x = 2$ and $x = 4$. What is the increase in kinetic energy (in joules) between these positions?
34. The engine in Fig. 9.R.2 is using energy at a rate of 300 joules per second to lift the weight of 600 kilograms. If the engine operates at 60% efficiency, at what speed (meters per second) can it raise the weight?

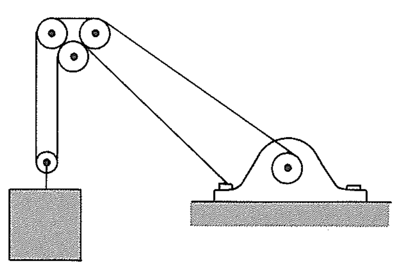


Figure 9.R.2. The engine for Problem 34.

35. Find a formula for the work required to empty a tank of water which is a solid of revolution about a vertical axis of symmetry.
36. How much work is required to empty the tank shown in Fig. 9.R.3? [*Hint:* Use the result of Exercise 35.]

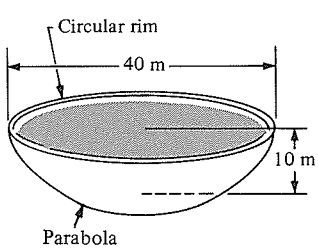


Figure 9.R.3. How much energy is needed to empty the tank?

37. The pressure (force per unit area) at a depth h below the surface of a body of water is given by $p = \rho g h = 9800h$, measured in newtons per square meter. (This formula derives from the fact that the force needed to support a column of water of cross-sectional area A is (volume) $\times$ (density) $\times$ $(g) = Ah\rho g$, so the force per unit area is $\rho g h$, where $\rho = 10^3$ kilograms per cubic meter, and $g = 9.8$ meters per second per second).
    (a) For the dam shown in Fig. 9.R.4(a), show that the total force exerted on it by the water is $F = \frac{1}{2}\int_a^b \rho g[f(x)]^2\,dx$. [*Hint:* First calculate the force exerted on a vertical rectangular slab.]
    (b) Make up a geometric theorem relating F to the volume of a certain solid.
    (c) Find the total force exerted on the dam whose face is shown in Fig. 9.R.4(b).

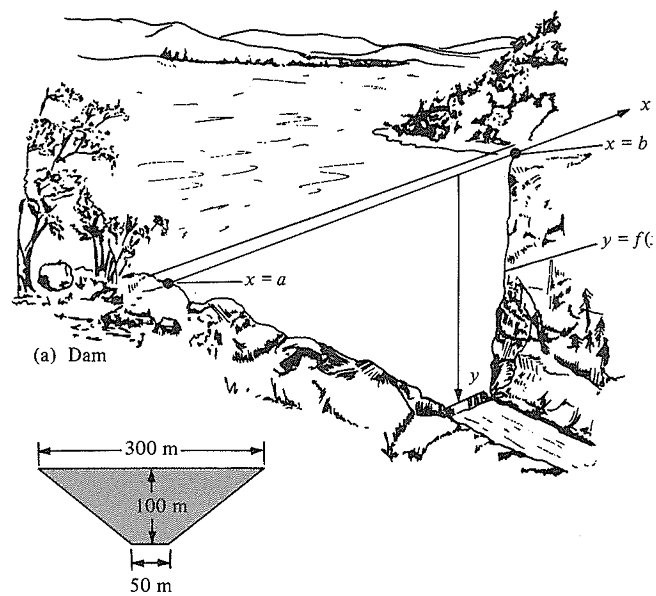


Figure 9.R.4. Calculate the force on the dam.

38. (a) *Pappus' theorem for volumes.* Use the shell method to show that if a region R in the xy plane is revolved around the y axis, the volume of the resulting solid equals the area of R times the circumference of the circle obtained by revolving the center of mass of R around the y axis.
    (b) Use Pappus' theorem to do Exercise 21(a) in Section 9.2.
    (c) Assuming the formula $V = \frac{4}{3}\pi r^3$ for the volume of a ball, use Pappus's theorem to find the center of mass of the semicircular region $x^2 + y^2 \leqslant r^2$, $x \geqslant 0$.

★39. See the instructions for Exercises 17–22.
    (a) Suppose that $f(t)$ is a step function on $[a, b]$, with value k_i on the interval (t_{i-1}, t_i) belong-

ing to a partition $(t_0, t_1, \ldots, t_n)$. Find a formula for the standard deviation of f on $[a,b]$.

(b) Simplify your formula in part (a) for the case when all the Δt_i's are equal.

(c) Show that if the standard deviation of a step function is zero, then the function has the same value on all the intervals of the partition; i.e., the function is constant.

(d) Give a definition for the standard deviation of a list $a_1, \ldots, a_n$ of numbers.

(e) What can you say about a list of numbers if its standard deviation is zero?

★40. (a) Prove, by analogy with the mean value theorem for integrals, the *second mean value theorem*: If f and g are continuous on $[a,b]$ and $g(x) \geqslant 0$, for x in $[a,b]$, then there is a point t_0 in $[a,b]$ such that

$$\int_a^b f(t)g(t)\,dt = f(t_0)\int_a^b g(t)\,dt.$$

(b) Show that the mean value theorem for integrals is a special case of the result in part (a).

(c) Show by example that the conclusion of part (a) is false without the assumption that $g(t) \geqslant 0$.

★41. Show that if f is an increasing continuous function on $[a,b]$, the mean value theorem for integrals implies the conclusion of the intermediate value theorem.

★42. Show that in the context of Exercise 35, the work needed to empty the tank equals Mgh, where M is the total mass of water in the tank and h is the distance of the center of mass of the tank below the top of the tank.

★43. Let $f(x) > 0$ for x in $[a,b]$. Find a relation between the average value of the logarithmic derivative $f'(x)/f(x)$ of f on $[a,b]$ and the values of f at the endpoints.